

UNDERSTANDING DISCRETE ROTATIONS

Michael S. Richman

Center for Applied Mathematics
Cornell University
Ithaca, NY 14853

Thomas W. Parks

Department of Electrical Engineering
Cornell University
Ithaca, NY 14853

ABSTRACT

The concept of rotations in continuous-time, continuous-frequency is extended to discrete-time, discrete-frequency as it applies to the Wigner distribution. As in the continuous domain, discrete rotations are defined to be elements of the special orthogonal group over the appropriate (discrete) field. Use of this definition ensures that discrete rotations will share many of the same mathematical properties as continuous ones. A formula is given for the number of possible rotations of a prime-length signal, and an example is provided to illustrate what such rotations look like. In addition, by studying a 90 degree rotation, we formulate an algorithm to compute a prime-length discrete Fourier transform (DFT) based on convolutions and multiplications of discrete, periodic chirps. This algorithm provides a further connection between the DFT and the discrete Wigner distribution based on group theory.

1. INTRODUCTION

The Wigner distribution satisfies many desirable properties, among them being the property of maximal covariance [1, 2]. This property results from the relationship between the Wigner distribution and the Weyl correspondence, as the Weyl correspondence is the unique correspondence that is well-behaved under symplectic transformations in the following sense: applying a change of coordinates to a Wigner distribution of a signal is identical to computing the Wigner distribution of that signal after an appropriate combination of dilations, shearings, and rotations has been applied to it, [3, 4]. These concepts are completely straightforward for the continuous Wigner distribution, and have been detailed elsewhere, [5]. In particular, the concept of applying a rotation to a continuous time-frequency distribution is easily understood. To rotate a distribution by an angle θ , simply apply the matrix $R(\theta)$, given

by

$$R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad (1)$$

to the coordinates of the Wigner distribution. In the time domain, this is equivalent to applying the appropriate fractional Fourier transform, [6]. The case of $\theta = -\frac{\pi}{2}$ is particularly significant, as this corresponds in the time domain to a Fourier transform. In [7], Poletti gave a time-frequency interpretation of how this rotation could be decomposed into a series of chirps, which had been described previously in the context of the ambiguity function, [8].

For the case of discrete-time, discrete-frequency time-frequency distributions, rotations are more difficult to understand because the time-frequency plane is periodic and has finitely many points. Until recently, there was not a formulation of the Wigner distribution in the signal processing literature that could be related to the discrete Weyl correspondence. In [9, 10], a discrete-time, discrete-frequency Wigner distribution was derived that does satisfy the Weyl correspondence for this discrete domain. Since it is clear that the discrete-time, discrete-frequency domain is different from the continuous-time, continuous-frequency domain, [11, 12], it is not surprising that the interpretation of rotations is different as well. In this paper, the concept of rotations will be generalized to discrete-time, discrete-frequency using an algebraic construction. We will also describe how a 90 degree rotation in discrete-time, discrete-frequency can be decomposed into a sequence of discrete, periodic chirps.

2. DISCRETE ROTATIONS

In [10], the discrete-time, discrete-frequency Wigner distribution was shown to satisfy the property of covariance, made possible by its relationship with the Weyl correspondence. Specifically, the application of a symplectic transformation, $\mathcal{A}$, to a Wigner distribution, $W_{x,y}$, is equivalent to first applying a unitary

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transformation dependent on $\mathcal{A}, \mathcal{U}(\mathcal{A})$, to both x and y , and then computing their Wigner distribution, i.e.

$$W_{x,y}(\mathcal{A}(t, f)) = W_{\mathcal{U}(\mathcal{A})x, \mathcal{U}(\mathcal{A})y}(t, f). \quad (2)$$

In other words, if a linear transformation with unit norm is applied to a Wigner distribution in the time-frequency plane, there is a corresponding transformation of the signals that yields the same Wigner distribution.

While there are three fundamental types of symplectic transformations (dilation/compression, shearing, and rotations), the focus here is on rotations. The only rotation explored in [10] was the symplectic transformation $\mathcal{A}_\gamma$ corresponding to rotation counterclockwise by 90 degrees, given by

$$\mathcal{A}_\gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (3)$$

and the corresponding unitary transformation $\mathcal{U}(\mathcal{A}_\gamma)$ for a length- N signal $x(n)$ such that (2) is satisfied is given by

$$\mathcal{U}(\mathcal{A}_\gamma)x(n) = j^{\frac{1}{2}} \sum_{k=0}^{N-1} e^{j\frac{2\pi nk}{N}} X(k), \quad (4)$$

where $X(k)$ is the discrete Fourier transform (DFT) of $x(n)$. $\mathcal{U}(\mathcal{A}_\gamma)$ is proportional to the inverse DFT. Figure 1a contains an image plot of the discrete Wigner distribution of the signal $x_1(n)$ given in *MATLAB* by $x_1(n) = \text{ones}(17, 1)$. Figure 1b contains an image plot of the discrete Wigner distribution of the signal $\mathcal{U}(\mathcal{A}_\gamma)x_1(n)$. Alternatively, figure 1b contains the result of applying $\mathcal{A}_\gamma$ to the Wigner distribution given in figure 1a. As shown in figure 1b, application of the symplectic transformation $\mathcal{A}_\gamma$ does result in a rotation by 90 degrees.

Consider now other rotations in discrete-time, discrete-frequency. To simplify matters, we limit our discussion to signals whose lengths are prime. In the continuous time-frequency plane, a rotation by θ is given by the matrix in (1). The set of such rotation matrices in two dimensions comprise an algebraic group called the special orthogonal group of dimension 2 over the real numbers, or $SO(2, \mathcal{R})$. (This group is discussed in numerous books; see, for example, [13]). The elements $R(\theta)$ of $SO(2, \mathcal{R})$ satisfy the following properties, [6]:

1. Zero rotation: $R(0) = I$
2. Consistency with Fourier transform: $R(-\frac{\pi}{2}) = \mathcal{F}$
3. Additivity of rotations: $R(\alpha)R(\beta) = R(\alpha + \beta)$

In discrete-time, discrete-frequency for a length- p signal (p prime), let a rotation matrix be any element in the analogous group of matrices, the special orthogonal group of dimension 2 over the integers modulo p , $SO(2, \mathcal{Z}/p)$. Then, every element R_p of $SO(2, \mathcal{Z}/p)$ has the form, [14],

$$R_p(a, b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad (5)$$

where a, b are elements of the integers mod p , $\mathcal{Z}/p$, and the determinant of $R_p(a, b)$ is equal to 1, i.e.

$$a^2 + b^2 = 1 \pmod{p}. \quad (6)$$

Note that such matrices have a similar form to that given in (1). Furthermore, since the identity matrix and $\mathcal{A}_\gamma^{-1}$ are both elements of $SO(2, \mathcal{Z}/p)$, this definition satisfies the first two properties cited from [6]. The third property is satisfied by the fact that $SO(2, \mathcal{Z}/p)$ has a generator, [14], and so every element can be expressed as a power of the generator (multiplication of elements is accomplished by the addition of exponents).

Another consequence is that there are only finitely many rotations possible for any given p . To obtain the exact number for $p > 2$ ($p = 2$ is a trivial case), first express p as $p = 4l \pm 1$. Then, the number of possible rotations, i.e. number of elements of $SO(2, \mathcal{Z}/p)$, is given by, [14],

$$\text{Number of rotations} = \begin{cases} p-1 & \text{if } p = 4l+1, \\ p+1 & \text{if } p = 4l-1. \end{cases} \quad (7)$$

An example illustrates how this works in practice. For the signal $x_1(n) = \text{ones}(17, 1)$, the group $SO(2, \mathcal{Z}/p)$ has 16 elements. A generator of the group $SO(2, \mathcal{Z}/17)$, R_{g17} , is given by

$$R_{g17} = \begin{bmatrix} 5 & -8 \\ 8 & 5 \end{bmatrix}. \quad (8)$$

It can be verified that $R_{g17}^4 = R_{17}(\frac{\pi}{2})$ and $R_{g17}^{16} = R(0) = I$. The rotation of x_1 by R_{g17}^4 (90 degrees) is shown in figure 1b. The rotation by R_{g17}^2 (45 degrees) is shown in figure 1c, and the rotation by R_{g17} is shown in figure 1d. The interpretation of figure 1d is difficult, as the elements around the origin (the origin is fixed by R_{g17}) have been permuted in a manner which does not lend itself to simple explanation. While these rotations are being depicted in the time-frequency plane, they can be computed directly in the time domain, by expressing a rotation matrix as a product of shearing and 90 degree rotation matrices, and then applying the corresponding sequence of time domain transformations,

i.e. the product

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \mathcal{A}_\gamma^{-1} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \mathcal{A}_\gamma \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad (9)$$

has the corresponding time domain operator

$$e^{\frac{j2\pi(2^{-1}\alpha n^2)p}{p}} \mathcal{F} e^{\frac{j2\pi(2^{-1}(-b)n^2)p}{p}} \mathcal{F}^{-1} e^{\frac{j2\pi(2^{-1}\alpha n^2)p}{p}}, \quad (10)$$

where $\alpha = (a-1)b^{-1}$, $\mathcal{F}$ refers to the DFT, and $(\cdot)_p$ refers to arithmetic modulo p .

It is possible, albeit more complicated, to generalize the results here to non-prime lengths. Such complications arise from the fact that $\mathbb{Z}/n$ is harder to characterize when n is not prime. Based on the discussion presented here, there is some justification to consider (10) as a definition for the discrete fractional Fourier transform. This differs from definitions given in [15, 16]. Due to the difficulties in interpreting some of the rotations (e.g. figure 1d) obtained from (10), it would be premature to make any further claims about its relationship to the discrete fractional Fourier transform. This analysis, though, might provide insight into why it is difficult to make a suitable definition of this transform.

3. COMPUTING THE DFT VIA PERIODIC CHIRPS

It is well-known that the DFT can be computed via a sequence of chirps, as given in [17]. We now derive a similar formula using discrete, periodic chirps (like those in (10)). This result provides an interesting viewpoint of the relationship between the discrete-time, discrete-frequency Wigner distribution and the DFT. It also may have some potential use with regards to applications of linear swept frequency measurements for periodic systems, in a manner analogous to what was given in [7]. The following discussion will assume that the signal of interest is of length- p , a prime, although, as before, the results may be generalized to other lengths.

The rotation matrix given by $R_p(-\frac{\pi}{2})$, as stated previously, corresponds to the DFT. This matrix can be written as a product of three shearing matrices:

$$R_p(-\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \quad (11)$$

This corresponds to the following equation in the time domain for the length- p signal, $x(n)$:

$$X(m) = [x(n) \circledast e^{\frac{j2\pi(2^{-1}n^2)p}{p}}] e^{\frac{-j2\pi(2^{-1}n^2)p}{p}} \circledast e^{\frac{j2\pi(2^{-1}n^2)p}{p}}, \quad (12)$$

where $\circledast$ denotes circular convolution.

Equation (12) gives an algorithm for a prime-length DFT that uses these unusual periodic chirps. Figures 2 and 3 illustrate just how these chirps differ from samples of continuous chirps. Figure 2 contains a plot of the real and imaginary parts of the standard chirp signal $x_2(n) = e^{\frac{-j\pi n^2}{17}}$, $n = -8, \dots, 8$. Figure 3 contains a plot of the real and imaginary parts of the periodic chirp signal $x_3(n) = e^{\frac{-j2\pi(2^{-1}n^2)_{17}}{17}}$ on the same interval. With such chirps, the linear swept frequency measurement and the chirp transform can be formulated for discrete, periodic domains.

4. CONCLUSION

An extension of the concept of rotation to discrete-time, discrete-frequency has been formulated using well-known groups from algebra. The rotation operators obtained satisfy several desired mathematical properties. Unfortunately, it is not always straightforward to interpret the effect of such operators in the time-frequency plane. Since a satisfactory definition of the discrete fractional Fourier transform remains an open question, it may be important to consider details gleaned from analysis of these discrete rotations, such as the fact that there are only finitely many of them for a given signal length. It was also demonstrated how the study of 90 degree rotations of the discrete Wigner distribution leads to an alternative algorithm for a prime-length DFT. While this does not directly lend itself to any applications, further study of the discrete, periodic chirps utilized in the algorithm may prove useful in the analysis of periodic systems.

5. REFERENCES

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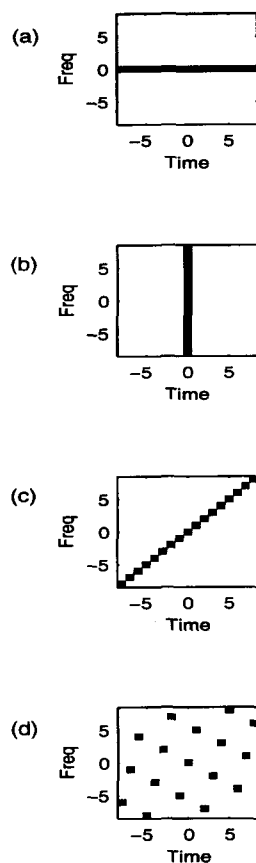


Figure 1: The discrete Wigner distribution of:

- (a) the signal $x_1(n) = \text{ones}(17, 1)$,
- (b) the rotation Rg_{17}^4 (90 degrees) applied to $x_1(n)$,
- (c) the rotation Rg_{17}^2 (45 degrees) applied to $x_1(n)$,
- (d) the rotation Rg_{17} applied to $x_1(n)$.

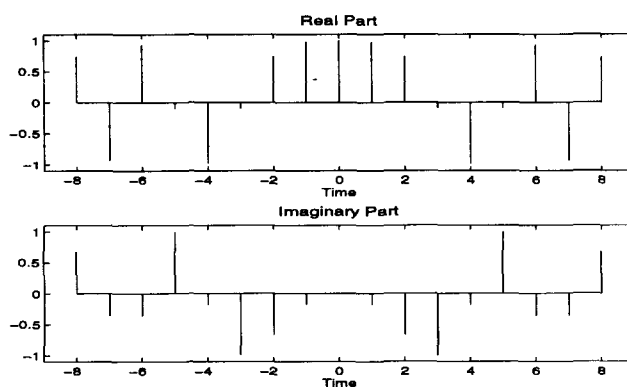


Figure 2: The standard chirp $x_2(n) = e^{\frac{-j\pi n^2}{17}}$.

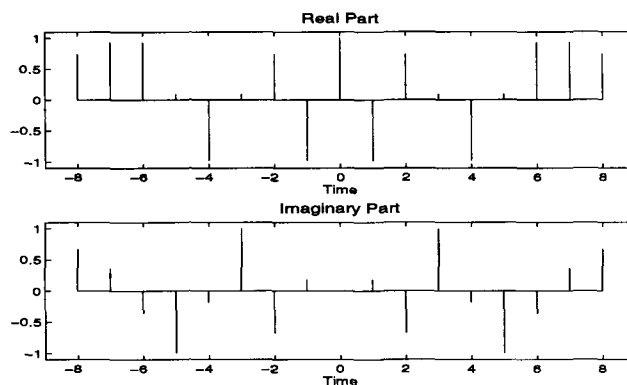


Figure 3: The periodic chirp $x_3(n) = e^{\frac{-j2\pi(2^{-1}n^2)_{17}}{17}}$.