DESIGN OF PARAUNITARY OVERSAMPLED COSINE-MODULATED FILTER BANKS

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ABSTRACT

In this paper we derive perfect reconstruction (PR) conditions for oversampled cosine-modulated filter banks. The results can be regarded as a generalization of the known work for critical subsampling. We show that in the oversampled case we gain some additional degree of freedom, which can be exploited in the filter design process. This leads to PR prototypes with stopband attenuations being much higher than in the critically subsampled PR case. The filters designed as PR filters for the oversampled case can also serve as prototypes for critically subsampled cosine-modulated pseudo QMF banks.

1. INTRODUCTION

Applications of oversampled filter banks can be found in those areas of signal processing where one is interested in making modifications to signals in certain frequency bands. Examples are the simulation of room acoustics by filtering in subbands, noise reduction in the spectral domain, and equalization via fixed or dynamic (i.e. time varying) filtering of subband signals. On the other hand, critically subsampled filter banks are very attractive for subband coding, because they lead to a minimum number of transmitted coefficients. But they are not useful, if we simply want to implement some linear filtering by introducing gain factors in the subbands, because the main aliasing components will not cancel at the output.

Recently, perfect reconstruction (PR) conditions for oversampled DFT filter banks have been derived, and general relations between oversampled filter banks and frame theory have been investigated [1–3]. In this paper we consider PR conditions for oversampled cosine-modulated filter banks, because due to the real-valued subband signals, these filter banks are more suitable for spectral modification than oversampled DFT filter banks.

2. POLYPHASE REPRESENTATION

The analysis polyphase matrix of a filter bank is defined as

$$[\mathbf{E}^{(L)}(z)]_{kj} = H_{kj}(z),\ k = 0,\dots,M-1,\ j = 0,\dots,N-1,$$

where the polyphase components $H_{kj}(z)$ of the k-th analysis filter $h_k(n)$ are given as

$$H_{kj}(z) = \sum_{\ell=-\infty}^{\infty} h_k(\ell N + j)\ z^{-\ell}.$$

M denotes the number of subbands and N the subsampling factor in each subband. The oversampling ratio $L = M/N$ is restricted to be an integer.

For critical subsampling we obtain a quadratic polyphase matrix $E^{(1)}(z)$ of size $M \times M$. The polyphase analysis filter bank corresponding to this case is depicted in Figure 1(a) for $M = 4$. In the oversampled case the polyphase matrix $E^{(L)}(z)$ has a rectangular shape of size $M \times N$. The corresponding polyphase analysis filter bank is shown in Figure 1(b) for $M = 4$ and $N = 2$.

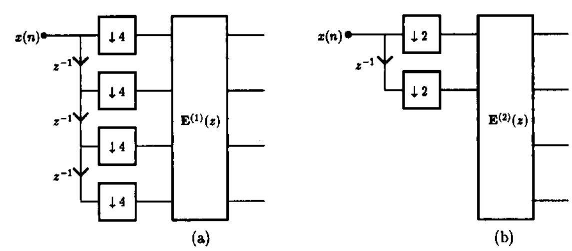

Figure 1: Polyphase analysis filter bank ($M = 4$): (a) Critically subsampled, (b) oversampled by factor 2.

Let us consider a polyphase matrix $\mathbf{E}^{(1)}(z)$ for the critically sampled case ($L = 1$). When we now increase the subband sampling rate by a factor $L \geq 2$, the new analysis polyphase matrix $\mathbf{E}^{(L)}(z)$ can be obtained according to

$$\mathbf{E}^{(L)}(z) = \mathbf{E}^{(1)}(z^L) \cdot \mathbf{S}(z) \qquad (1)$$

with $\mathbf{S}(z) = [\mathbf{I}_N, z^{-1}\mathbf{I}_N, z^{-2}\mathbf{I}_N, \cdots, z^{-(L-1)}\mathbf{I}_N]^{\mathrm{T}} \in \mathbb{R}^{M\times N}$, $\mathbf{I}_N$ denoting the $N \times N$ identity matrix.

3. COSINE-MODULATED FILTER BANKS WITH ARBITRARY SUBSAMPLING RATE

We consider cosine-modulated filter banks where the analysis and synthesis filters $h_k(n)$ and $f_k(n)$, resp., are derived from a linear-phase prototype $p(n)$ according to

$$h_k(n) = 2p(n)\cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(n-\frac{L_p-1}{2}\right)+(-1)^k\frac{\pi}{4}\right],$$
$$f_k(n) = h_k(L_p-n-1),$$

and $n=0,\ldots,L_p-1,\ k=0,\ldots,M-1$. We restrict us to even M and prototype lengths of $L_p=2mM,\ m\in\mathbb{N}$.

3.1. Perfect Reconstruction Conditions for the Critically Sampled Case [4]

Let $P_j(z),\ j=0,\ldots,2M-1$, denote the polyphase components of the linear-phase prototype $p(n)$ according to

$$P_j(z)=\sum_{\ell=0}^{m-1} p(2\ell M+j)\ z^{-\ell}.$$

The polyphase matrix can be written as

$$\mathbf{E}^{(1)}(z)=\mathbf{T}\begin{bmatrix}\mathbf{p}_0(z^2)\\ z^{-1}\mathbf{p}_1(z^2)\end{bmatrix}=\mathbf{T}\ \mathbf{P}(z^2) \qquad (2)$$

where

$$[\mathbf{T}]_{kj}=2\cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(j-\frac{L_p-1}{2}\right)+(-1)^k\frac{\pi}{4}\right], \qquad (3)$$
$$\text{with}\quad k=0,\ldots,M-1,\quad j=0,\ldots,2M-1;$$

$$\mathbf{p}_0(z^2) = \operatorname{diag}\left[P_0(-z^2),P_1(-z^2),\ldots,P_{M-1}(-z^2)\right],$$
$$\mathbf{p}_1(z^2) = \operatorname{diag}\left[P_M(-z^2),P_{M+1}(-z^2),\ldots,P_{2M-1}(-z^2)\right].$$

Perfect reconstruction is achieved if

$$\tilde{\mathbf{E}}^{(1)}(z)\ \mathbf{E}^{(1)}(z)=\mathbf{I}_M, \qquad (4)$$

where $\tilde{\mathbf{E}}(z)=\mathbf{E}_*^{\mathrm{T}}(z^{-1})$ and $*$ denotes complex conjugation of the filter coefficients. Then the polyphase components $P_j(z)$ satisfy the following constraint:

$$\tilde{P}_k(z)P_k(z)+\tilde{P}_{k+M}(z)P_{k+M}(z)=\frac{1}{2M} \qquad (5)$$
$$\text{for}\quad k=0,\ldots,\frac{M}{2}-1.$$

3.2. Perfect Reconstruction Conditions for the Oversampled Case

The polyphase matrix for the oversampled case can be obtained by inserting (2) into (1) which yields

$$\mathbf{E}^{(L)}(z)=\mathbf{T}\ \mathbf{P}(z^{2L})\ \mathbf{S}(z)=\mathbf{T}\begin{bmatrix}\mathbf{p}_0(z^{2L})\\ z^{-1}\mathbf{p}_1(z^{2L})\\ \vdots\\ z^{-(2L-1)}\mathbf{p}_{(2L-1)}(z^{2L})\end{bmatrix} \qquad (6)$$

with

$$\mathbf{p}_\ell(z^{2L})=\operatorname{diag}[P_{\ell N}(-z^{2L}),P_{\ell N+1}(-z^{2L}),\ldots,\ldots,P_{\ell N+(N-1)}(-z^{2L})] \qquad (7)$$

and $\ell=0,\ldots,2L-1$.

Now two cases arise:

(i) If the critically sampled filter bank has the perfect reconstruction property (4), then the corresponding oversampled filter bank obviously also guarantees perfect reconstruction and

$$\tilde{\mathbf{E}}^{(L)}(z)\ \mathbf{E}^{(L)}(z)=L\cdot\mathbf{I}_N. \qquad (8)$$

This can be easily seen from (1) and (4), because we have

$$\begin{aligned}\tilde{\mathbf{E}}^{(L)}(z)\ \mathbf{E}^{(L)}(z) &= \tilde{\mathbf{S}}(z)\ \tilde{\mathbf{E}}^{(1)}(z)\ \mathbf{E}^{(1)}(z)\ \mathbf{S}(z)\\ &= \tilde{\mathbf{S}}(z)\ \mathbf{S}(z)=L\cdot\mathbf{I}_N.\end{aligned}$$

(ii) When the L-times oversampled filter bank is designed in such a way that (8) is satisfied and (4) is not, perfect reconstruction is obtained for all oversampling ratios $L_1\geq L$, but the filter bank designed this way ensures only almost perfect reconstruction for all $L_2<L$. However, this is the interesting case for the design of oversampled PR filter banks, because the conditions on the prototype can be relaxed, and filters with higher stopband attenuation can be designed than in the first case.

For the latter case we show now that (8) can be satisfied for arbitrary oversampling ratios L. Inserting (6) into equation (8) yields the following condition:

$$\tilde{\mathbf{S}}(z)\ \tilde{\mathbf{P}}(z^{2L})\ \mathbf{T}^{\mathrm{T}}\mathbf{T}\ \mathbf{P}(z^{2L})\ \mathbf{S}(z)\overset{!}{=}L\cdot\mathbf{I}_N \qquad (9)$$

Choosing $\mathbf{T}$ as in (3) with $L_p=2mM$ leads to the following expression for the product $\mathbf{T}^{\mathrm{T}}\mathbf{T}$, where the upper sign is valid for even m and the lower for odd m, and $\mathbf{J}_M$ denotes the $M\times M$ reverse identity matrix [4]:

$$\mathbf{T}^{\mathrm{T}}\mathbf{T}=2M\cdot\begin{bmatrix}\mathbf{I}_M\mp\mathbf{J}_M & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_M\pm\mathbf{J}_M\end{bmatrix} \qquad (10)$$

Inserting this product into (9) we get

$$2M\cdot\tilde{\mathbf{S}}(z)\ \tilde{\mathbf{P}}(z^{2L})\mathbf{P}(z^{2L})\ \mathbf{S}(z)+2M\cdot\tilde{\mathbf{S}}(z)\ \tilde{\mathbf{P}}(z^{2L})\cdot \cdot\begin{bmatrix}\mp\mathbf{J}_M & \mathbf{0}\\ \mathbf{0} & \pm\mathbf{J}_M\end{bmatrix}\mathbf{P}(z^{2L})\ \mathbf{S}(z)\overset{!}{=}L\cdot\mathbf{I}_N. \qquad (11)$$

For achieving the PR property, the second part of this equation containing the antidiagonal terms must be zero, which can be written with (6) as

$$\mp\sum_{\ell=0}^{L-1} z^{(L-1)-2\ell}\ \tilde{\mathbf{p}}_{L-1-\ell}(z^{2L})\ \mathbf{J}_N\ \mathbf{p}_\ell(z^{2L})\ \pm \pm\sum_{\ell=0}^{L-1} z^{(L-1)-2\ell}\ \tilde{\mathbf{p}}_{2L-1-\ell}(z^{2L})\ \mathbf{J}_N\ \mathbf{p}_{L+\ell}(z^{2L})\overset{!}{=}\mathbf{0}. \qquad (12)$$

Since the prototype filter $p(n)$ is linear-phase and its length is assumed to be an integer multiple of $2M$, always two polyphase components are related as

$$P_j(z)=z^{-(m-1)}\tilde{P}_{2M-j-1}(z)\quad\text{for}\quad j=0,\ldots,M-1.$$

Applying this equation to every element of $\mathbf{p}_\ell(z^{2L})$ in (7) and replacing z by $-z^{2L}$ yields a similar relation for two polyphase diagonal matrices, namely

$$\mathbf{p}_\ell(z^{2L}) = z^{-2L(m-1)}\ (-1)^{m-1}\ \mathbf{J}_N\ \tilde{\mathbf{p}}_{2L-\ell-1}(z^{2L})\ \mathbf{J}_N,$$

$\ell = 0, \ldots, 2L-1$. Inserting this equation into the left side of (12) we see that the right side of (12) is indeed satisfied and the condition (11) reduces to

$$2M \cdot \tilde{\mathbf{S}}(z)\ \tilde{\mathbf{P}}(z^{2L})\ \mathbf{P}(z^{2L})\ \mathbf{S}(z) = L \cdot \mathbf{I}_N. \qquad (13)$$

This shows that the polyphase matrix for the oversampled case in (6) indeed satisfies the condition (8) with all transform matrices $\mathbf{T}$ having the property (10) for a fixed prototype length $L_p = 2mM$, $m \in \mathbb{N}$. We can exploit this fact by *directly* designing a prototype for an arbitrary oversampling ratio L instead of using a prototype for the critically subsampled case $L = 1$ or generally for $L_2 < L$. This extends the known results for critical subsampling summarized in section 2.1 in a general way.

Rewriting (13) into a condition for the polyphase components $P_j(z)$, $j = 0, \ldots, 2M-1$, yields

$$\sum_{\ell=0}^{2L-1} \tilde{P}_{k+\ell N}(z)\ P_{k+\ell N}(z) = \frac{1}{2N} \quad \text{for} \quad k = 0, \ldots, \left\lceil \frac{N}{2} \right\rceil - 1. \qquad (14)$$

The index k can be restricted to the first $\lceil N/2 \rceil$ rows of (13), because they are exactly the same as the last $N/2$ ones ($N \neq 1$). The case $N = 1$ has been taken into account by use of the ceiling-operator.

From (14) we can see that always $2L$ polyphase components have to be power-complementary in order to provide a perfect reconstruction linear-phase prototype $P(z)$ for the L-times oversampled filter bank. The critically sampled case in (5) requires pairwise power complementarity of the polyphase components, which is a more restrictive condition.

4. PROTOTYPE DESIGN

The fact that the requirements on the polyphase components in the oversampled case can be relaxed is directly reflected by the number of constraints used for coefficient optimization, which is a typical time domain operation. Consider eq. (14), where we have $\lceil N/2 \rceil$ frequency domain PR conditions to be satisfied. Inverse z-transform of (14) yields

$$\sum_{\ell=0}^{2L-1} p_{k+\ell N}(n) * p_{k+\ell N}(-n) = \frac{1}{2N} \cdot \delta(n),$$

where $\delta(n)$ denotes the unit sample sequence. This is equivalent to

$$\sum_{\ell=0}^{2L-1} \underbrace{\sum_{i=0}^{m-1} p_{k+\ell N}(i) \cdot p_{k+\ell N}(i+n)}_{r_{k+\ell N}(n)} = \frac{1}{2N} \cdot \delta(n)$$

for $k = 0, \ldots, \lceil N/2 \rceil - 1$ and $n = -(m-1), \ldots, (m-1)$. Here, $r_{k+\ell N}(n)$ stands for the autocorrelation of the polyphase impulse responses $p_{k+\ell N}(n) = \mathcal{Z}^{-1}\{P_{k+\ell N}(z)\}$. Since we consider a prototype filter of length $L_p = 2mM$, $m \in \mathbb{N}$, each polyphase component $p_\ell(n)$ has the length m, which results in a total convolution length of $2m-1$. Due to the even symmetry of the autocorrelation function, we only have to take the values $n = 0, \ldots, (m-1)$ into account. With

$$\sum_{\ell=0}^{2L-1} r_{k+\ell N}(n) = \sum_{i=0}^{m-1} r_k(i) \delta(n-i)$$

we obtain $\lceil N/2 \rceil m$ PR constraints in the time domain:

$$r_k(0) = \frac{1}{2N}, \quad r_k(1) = 0, \quad \ldots \quad r_k(m-1) = 0, \qquad (15)$$

and $k = 0, \ldots, \lceil N/2 \rceil - 1$.

For the design of the linear-phase prototype we use a modified quadratic-constraint algorithm [5]. Here we obtain $\lceil N/2 \rceil m$ quadratic constraints, where each $r_k(n)$ in (15) can be written as $r_k(n) = \mathbf{p}^{\mathrm{T}} \mathbf{Q}_{kn} \mathbf{p}$ with $\mathbf{p} = [p(0), \ldots, p(L_p/2-1)]^{\mathrm{T}}$ and some matrices $\mathbf{Q}_{kn}$ with indices n and k as given above. For details refer to [5].

These constraints are solved numerically under additional minimization of the prototype's stopband energy, which can be written as

$$E_s = \sum_{k=0}^{Q-1} w_k \cdot \int_{\Omega_{s_k}}^{\Omega_{s_{k+1}}} P_0^2(e^{j\Omega})\, d\Omega \stackrel{!}{=} \min. \qquad (16)$$

Here $P_0(e^{j\Omega})$ denotes the real-valued symmetric amplitude response and $[w_0, \ldots, w_{Q-1}]$ the weighting factors for the different frequency regions, which are characterized by the edge frequencies $[\Omega_{s_0}, \ldots, \Omega_{s_{Q-1}}, \Omega_{s_Q} = \pi]$. This can be expressed as a quadratic form with the vector $\mathbf{p}$ as stated above:

$$E_s = \sum_{k=0}^{Q-1} w_k \cdot \mathbf{p}^{\mathrm{T}} \mathbf{S}_k \mathbf{p} = \mathbf{p}^{\mathrm{T}} \mathbf{S} \mathbf{p}$$

$$\text{and} \quad \mathbf{S} = \sum_{k=0}^{Q-1} w_k \int_{\Omega_{s_k}}^{\Omega_{s_{k+1}}} \mathbf{c}(\Omega) \mathbf{c}^{\mathrm{T}}(\Omega)\, d\Omega,$$

$$\mathbf{c}(\Omega) = \left[\cos\left(\frac{L_p-1}{2}\Omega\right), \cos\left(\frac{L_p-3}{2}\Omega\right), \ldots, \cos\left(\frac{\Omega}{2}\right)\right]^{\mathrm{T}}.$$

Note that it is possible to find analytic expressions for the elements of $\mathbf{S}$.

As an example, a prototype filter of length $L_p = 256$ for $M = 16$ subbands with a transition bandwidth of $b = 0.058\pi$ is designed. The NAG Fortran library was used for the optimization process. Figure 2(a) shows the magnitude frequency response of a prototype designed for an oversampled filter bank with $L = 2$, while the frequency response in Figure 2(b) belongs to the critically sampled case. The oversampled case yields higher stopband attenuation, since we have only half the number of PR constraints compared to the critically sampled case.

5. RELATION TO PSEUDO QMF FILTER BANKS

Equation (14) can be used not only for designing perfect reconstruction L-times oversampled filter banks, but also in

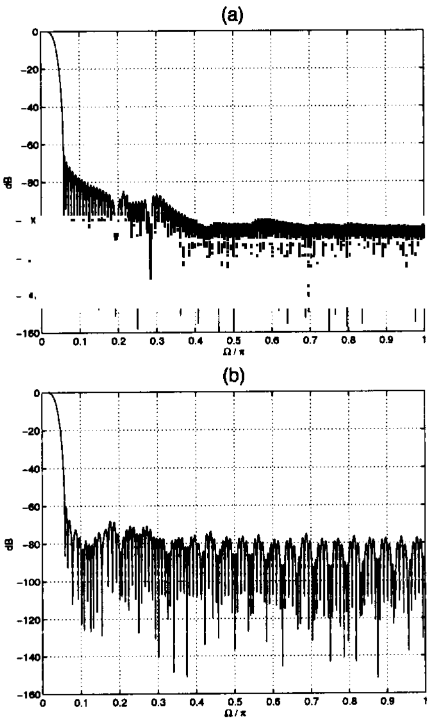

Figure 2: Magnitude frequency responses for $L_p = 256$ and $M = 16$: (a) Oversampled case with $L = 2$, (b) critically sampled case.

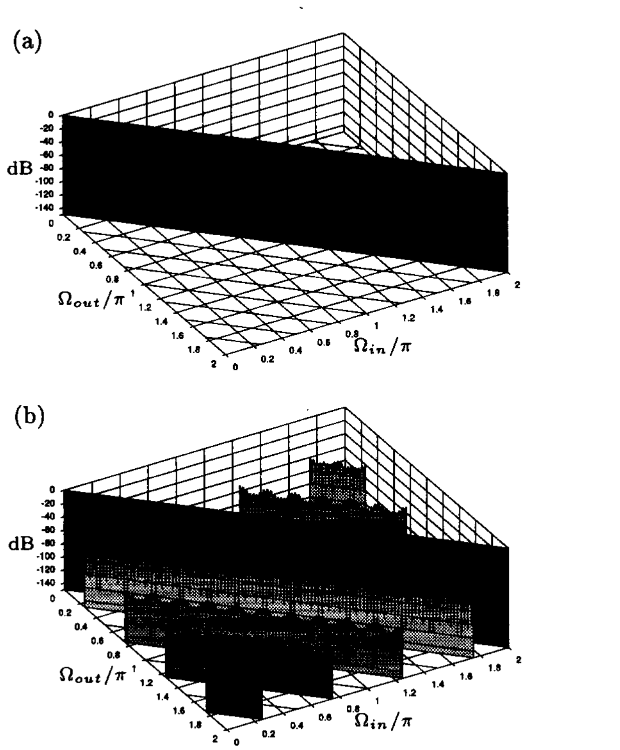

Figure 3: Magnitude bifrequency system function: (a) Oversampled perfect reconstruction case ($M = 8$, $L = 2$), (b) critically sampled pseudo QMF case.

order to obtain almost-PR solutions for oversampling ratios $L_1 < L$, where every L/L_1-th aliasing spectrum is canceled out.

An example is displayed in Figure 3, where measured magnitude bifrequency system functions [6] are used to visualize the aliasing distortions. Figure 3(a) shows the result for the perfect reconstruction oversampled case with $M = 8$ subbands and oversampling ratio $L = 2$. This prototype is now applied to a critically subsampled filter bank and the resulting bifrequency system function is shown in Figure 3(b). Note that in the pseudo QMF case we usually have $M - 1$ aliasing components. Because we have designed the prototype specially for $L = 2$, every second aliasing spectrum has disappeared.

6. CONCLUSION

The simplest way to design an oversampled PR modulated filter bank is to oversample a modulated filter bank that already gives PR for critical subsampling. However, as we showed in this paper, we have some additional degrees of freedom in the oversampled case, which yield prototypes for PR filter banks with stopband attenuations being much higher than in the case of critically subsampled PR filter banks. Here only $\lceil N/2 \rceil$ PR conditions have to be satisfied in order to design linear-phase prototypes compared to $M/2$ conditions in the critically sampled case. Moreover, we showed that prototype filters for L-times oversampled PR cosine modulated filter banks can be designed in such a way that they can serve as prototypes for critically sampled pseudo QMF banks, where every L-th aliasing spectrum is canceled out.

7. REFERENCES

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