UNIFIED APPROACH TO THE DESIGN OF QUADRATURE-MIRROR FILTERS

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ABSTRACT

A unified approach to the design of linear- and nonlinear-phase QMFs is developed. Formulated as an optimization problem, the design procedure is shown to translate into an eigenvalue-eigenvector problem. To find the optimal filter an algorithm is presented, which typically converges in a few tens of iterations. The flexibility of our design procedure permits several practical extensions to be made readily. These are (a) inclusion of frequency-weighted stopband energy criteria, and (b) inclusion of finite word-length constraints. We have successfully used our filters to applications such as image coding and analysis; here, their use in wavelet-series analysis of oceanographic data is demonstrated.

I. INTRODUCTION

The design of Quadrature Mirror Filters (QMF) has been widely studied in the literature in recent years [1],[2],[4]. Interest in their efficient design and application to subband coding of images is receiving renewed attention due in part to the application in wavelet-transform analysis of signals, images, and other systems. Related previous work on the design of two-channel QMF banks can be classified into a number of different approaches. One of the approaches consists in formulating the design problem as a constrained optimization problem. The resulting system can be made to exhibit close to PR behavior, while simultaneously achieving linear phase if desired. Another point in its favor is a self-contained design algorithm as will be shown in this paper. Finally, it permits flexibility in the design procedure so as to readily permit several practical extensions. These are (a) the inclusion of frequency-weighted stopband energy criteria [3], and (b) the *inclusion of finite word-length constraints*. The latter is stressed in the paper. We have successfully used our filters to applications such as image coding and analysis [5],[6]; here, their use in wavelet-series analysis of oceanographic data is demonstrated.

The organization of the paper is as follows. Section II discusses the QMF design problem in the framework of a two-channel filter-bank. Sections III and V describe the formulation of the nonlinear-phase and the linear-phase QMF design problems. In section IV an optimization algorithm is given; one of its merits is that it is self contained and does not rely on commercial optimization routines. Section VI indicates extensions alluded to earlier, focusing primarily on finite word-length constraints. In Section VII the application to multi-scale wavelet-series analysis of oceanographic data is discussed.

II. QMF FILTERBANK

Consider the two-channel QMF bank of Fig. 1, where all filters are assumed to be FIR, each with N taps. The reconstructed signal is readily shown to be

$$\hat{X}(z) = \frac{1}{2} [H_0(z) G_0(z) + H_1(z)G_1(z)] X(z)$$
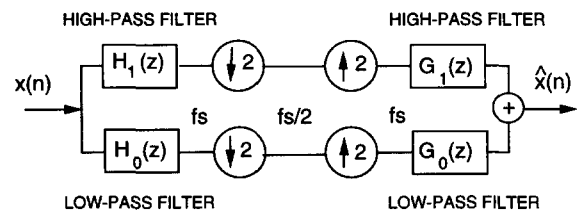$$+ \frac{1}{2} [H_0(-z) G_0(z) + H_1(-z) G_1(z)] X(-z)$$

Fig. 1 Two-channel QMF filterbank

The first term represents the desired time-invariant response, while the second term involving X(-z), is the aliasing term. The latter can be canceled by enforcing the relation $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$. In what follows, we develop expressions for the reconstruction-ripple and stopband energies for this filterbank.

III. FORMULATION OF NONLINEAR-PHASE QMF DESIGN PROBLEM

The discussion in this section is after [7]. For a QMF with an arbitrary number of taps N, we use the Smith-Barnwell QMF structure [8] with

$$H_0(z) = z^{-(N-1)} G_0(z^{-1})$$
$$H_1(z) = (-1)^{N-1} G_0(-z)$$
$$G_1(z) = (-z)^{-(N-1)} G_0(-z^{-1})$$

Note that we are taking the low-pass synthesis filter to be the prototype filter. Assume that the analysis filters are known, while the synthesis filters are unknown. Then

$$H_0(z) = z^{-(N-1)} \sum_{n=0}^{N-1} \gamma(n) z^n$$
$$H_1(z) = (-1)^{N-1} \sum_{n=0}^{N-1} (-1)^n \gamma(n) z^{-n}$$

$$G_0(z) = \sum_{n=0}^{N-1} g(n) z^{-n}$$

$$G_1(z) = (-1)^{N-1} \sum_{n=0}^{N-1} (-1)^n g(n) z^{(n-N+1)}$$

where g(n) are unknown filter weights, and $\gamma(n)$ denotes the values of these same filter tap coefficients obtained from the last iteration. It can then be shown that the reconstruction ripple energy is

$$E_r = g^T A g$$

The matrix A is given by

$$A(i, k) = 2 \sum_{n=0}^{[N/2-1]} a_{2n}(i) a_{2n}(k)$$

where

$$a_{2n}(k) = u(k-2n)\gamma(k-2n) + [u(k) - u(k-(N-1-2n))]\gamma(k+2n)$$

with u(•) denoting the unit step sequence. Further,

$$g = [g(0)\, g(1)\, \ldots\, g(N-1)]^T$$

Example: Consider the simple case where N=4, and γ = [0.1 0.5 0.5 0.7]. Then

$$A = \begin{bmatrix} 0.49 & 0.35 & 0.07 & 0.35 \\ 0.35 & 0.25 & 0.05 & 0.25 \\ 0.07 & 0.05 & 0.01 & 0.05 \\ 0.35 & 0.25 & 0.05 & 0.25 \end{bmatrix}$$

On the other hand, the stopband energy can be computed as

$$E_{sb} = \frac{1}{\pi} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} g(i)\, g(k) \int_{\omega_{sb}}^{\pi} \cos(i-k)\, d\omega = g^T F g$$

where the entries of the matrix F are

$$f_{i,k} = \begin{cases} \dfrac{1}{\pi}(\pi - \omega_{sb})\,, & i = k \\[2mm] -\dfrac{\sin(i-k)\omega_{sb}}{\pi(i-k)}\,, & i \neq k \end{cases}$$

Note that both matrices A and F are $N \times N$ dimensional.

IV. OPTIMIZATION ALGORITHM

The design problem can now be cast as an optimization problem. Minimize

$$E = E_r + \alpha E_{sb} = g^T A g + \alpha\, g^T F g$$

subject to $g^T g = 1$. Here, α is a stopband energy weighting parameter. The solution to above optimization problem is the eigenvector of $A + \alpha F$ corresponding to the minimum eigenvalue λ_{min} [2]. An iterative procedure (the Jain-Crochiere algorithm [2],[7]) is given in Appendix A.

Att (dB): attenuation at the first sidelobe in the stopband
r (dB):   maximum reconstruction ripple
df: Transition bandwidth = $f_{AT} - f_{3dB}$

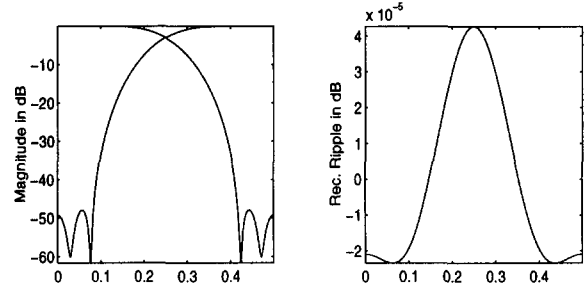An example design [7] for N=8 is shown in Fig. 2. Note that Att=48 dB, r=0.00004 dB.

Fig. 2   An 8-tap QMF design. Note that Att=48 dB, and reconstruction ripple = 0.00004 dB.

V. SIMPLIFICATION FOR LINEAR PHASE DESIGN

For the linear phase case, the complexity can be reduced. We will assume that the number of taps is even, i.e., N=2M where M is an integer. The structure used is the Croisier, Esteban, and Galand structure:

$$H_0(z) = H(z);\qquad G_0(z) = H(z);$$
$$H_1(z) = H(-z);\qquad G_1(z) = -H(-z);$$

where H(z), the prototype filter, is the low-pass analysis filter and is symmetric. Then it be shown that the ripple energy is given by

$$E_r = 2\, w^T A\, w$$

where w is the packed even-indexed part of h = [h(0), ..., h(N-1)]; i.e.,

$$w = [h(0)\, h(2)\, h(4)\, \ldots\, h(N-2)]$$

The matrix A is given by [2]

$$A(i, k) = \sum_{n=0}^{M-1} a_n(i)\, a_n(k)$$

where

$$a_n(k) = u(n-1-k)\,\eta(2(n+k)-N) + u(n+k-M)\,\eta(2(n-k)-1)$$

On the other hand, the stopband energy is given by

$$E_{sb} = 2\, v^T \hat{F}\, v = 2\, w^T F\, w$$

where the i,k entry of $\hat{F}$ is

$$\hat{f}_{i,k} = \frac{1}{\pi}\left(p_{i,k} + q_{i,k}\right)$$

$$p_{i,k} = \begin{cases} (\pi - \omega_{sb}) , & i = k \\ -\dfrac{\sin(i-k)\omega_{sb}}{\pi(i-k)} , & i \neq k \end{cases}$$

$$q_{i,k} = \begin{cases} (\pi - \omega_{sb}) , & i+k = N-1 \\ -\dfrac{\sin(i+k-(N-1))\omega_{sb}}{\pi(i+k-(N-1))} , & i+k \neq N-1 \end{cases}$$

The vector $\underline{v}$ denotes the first half of the vector $\underline{h}$, i.e.,

$$\underline{v} = [h(0)\ h(1)\ h(2)\ \ldots\ h(\tfrac{N}{2}-1)]$$

and is related to $\underline{w}$ through a suitable permutation matrix (recall that $\underline{h}$ is symmetric), as are also the matrices $\hat{F}$ and F.

Note that both matrices A and F are now $(N/2) \times (N/2)$ dimensional. The optimization problem essentially remains the same as for the nonlinear-phase filter design in Section IV, except that the unknown vector is now $\underline{w}$ which is $N/2$ dimensional. The computational complexity is accordingly reduced. Indeed, we now minimize

$$E = E_r + \alpha E_{sb} = \underline{w}^T A \underline{w} + \alpha\ \underline{w}^T F \underline{w}$$

subject to the constraint $2\underline{w}^T \underline{w} = 1$.

VI. EXTENSIONS

We have already described the extension to frequency-weighted criteria in [3]. Here, we discuss the extension of the technique to include finite word-length constraints. These constraints are readily incorporated by including a quantization step in the feedback loop of the algorithm. This is shown in Fig. 3 by the shadowed box (see also Appendix A). As a practical matter,
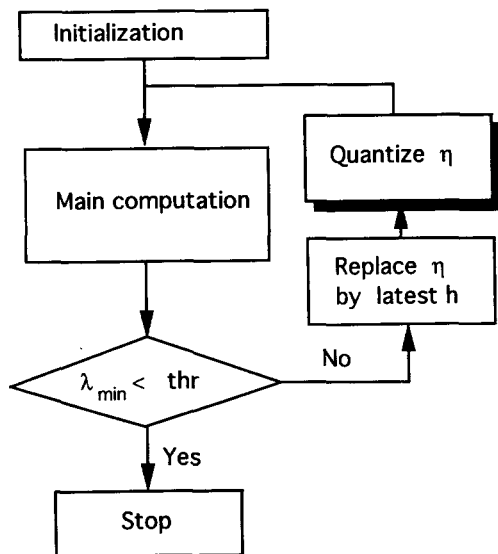

Fig. 3 Filter design algorithm incorporating the quantization step

however, the quantization constraint is imposed gradually, i.e., as the iterations increase, the number of quantization bits used (in the shadowed box) is decreased toward the target number of bits.

Design Examples

Consider the design of an 8-tap linear phase QMF filter H(z). The following parameters are used: $f_{sb} = 0.35$, $\alpha = 1.0$. Given below are the resulting filter coefficients for various number of bits. Such small number of bits are useful in video and other high-speed applications.

Bits	h0* 1024	h1* 1024	h2* 1024	h3* 1024	r dB	Att dB
4	0	64	704	- 64	0.07	-
6	16	96	704	-96	0.05	32
8	12	100	712	-100	0.04	32
10	13	99	710	-101	0.03	33

As a second example, consider the design of a 12-tap linear phase QMF filter H(z). Again, the following parameters are used: $f_{sb} = 0.35$, $\alpha = 1.0$. Given below are the resulting filter coefficients for 10 bits:

$$1024 \ast h(0{:}5) = [\,-3 \quad 0 \quad 119 \quad 705 \quad -115 \quad 19\,]$$

The passband ripple is less than 0.02 dB, and the stopband attenuation is 47 dB. Lower passband ripple can be achieved by choosing a larger value of f_{sb}, albeit at the expense of wider transition bandwidth.

We have demonstrated the design of quantized filters for the case of the linear-phase QMFs. A completely analogous approach can be applied to the nonlinear-phase filters as well.

VII. WAVELET ANALYSIS OF OCEANOGRAPHIC DATA

The data was collected from five stationary platforms in the Pacific ocean. The locations of these platforms form a diamond, with three of them located on the equator with a 100 km spacing, and the remaining two are located 100 km north and south of the center one. Each platform has sensors located at 27 different depths, that are integer multiples of 10 m, i.e., 10 m, 20 m, 30 m, etc. At each depth the east-west velocity, call it u(p,d,t) is measured; likewise, the south-north velocity is measured, call it v(p,d,t). Here, p=1,2,3,4,5; d=10,20,...,270; and t is the time in increments of 1 hour. Each data record, with 9737 sample points thus contains a little more than a year's history.

Using a tree-structured QMF filterbank with 8-tap linear-phase filters, wavelet analysis of the 27 x 5 x 2 = 270 data variables was carried out. For brevity, we show the results of analysis for u(1, 30, :). Fig. 4(a) shows five-level analysis. In
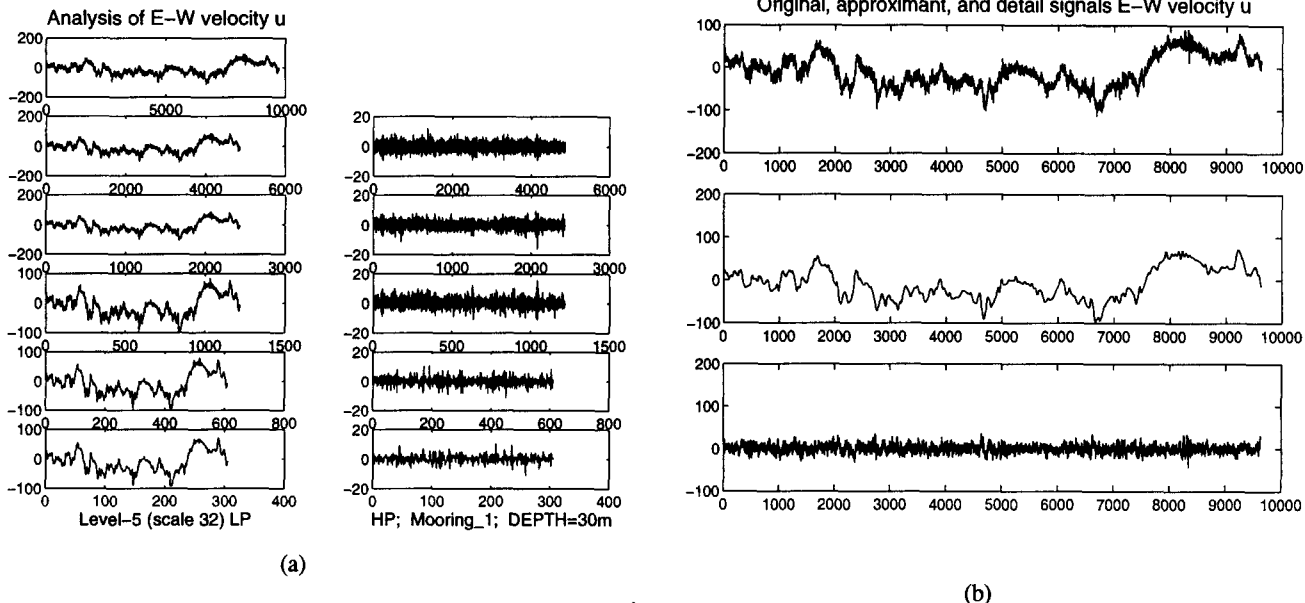
Fig. 4 Wavelet analysis of oceanographic data (from Pacific ocean). (a) Five-level, 32 scale, wavelet coefficients, (b) original, approximant, and detail

Fig. 4(b), the top trace represents the original data. In the middle is shown the the approximation signal reconstructed from the Level-5 (or 2^5-th scale) wavelet coefficients signal through five stages of upsampling and filtering. Finally, in the last trace of Fig. 4(b) is shown the difference between the original signal and the approximation signal, which may be called the detail.

CONCLUSIONS

We have presented a unified approach to the design of quadrature mirror filters. The approach is applicable to both linear and nonlinear phase QMF filters. Further, the formulation lends itself to an eigenvalue-eigenvector problem, for which we have developed a fast algorithm in the past. Here, we have extended the applicability of this design procedure to the case where the coefficients have limited precision. Such designs, with a small number of bits, are useful in video and other high-speed applications. Work is currently underway to extend the design technique to handle asymmetric designs; more specifically where the number of taps used at the encoder is larger than the number of taps at the decoder.

ACKNOWLEDGMENTS

APPENDIX A

Jain-Crochiere Algorithm:
This appendix presents an iterative algorithm for filter tap weight optimization. Refer to Fig. 3. The initializtion block consists of (a) setting, or reading in: N, ω_{sb}, and α, (b) calculating the matrix F as discussed in Section V, and (c) initializing η. The main computation block performs the following calculations for finding

the eigenvector of R corresponding to the minimum eigenvalue:
1. λ is initialized to $\lambda_0 = \underline{x}_0^T \underline{x}_0$
2. $\underline{z}_i = (R - \lambda_i I) \underline{x}_i$ (i is the current iteration index)
3. $d_i = \underline{z}_i^T \underline{x}_i$ and $c_i = \underline{z}_i^T \underline{z}_i$
4. $s_i = \lambda_i - (\underline{z}_i^T R \underline{z}_i) /c_i$
   $\gamma_i = [- s_i - (s_i^2 + 4 c_i + 4 d_i s_i)^{1/2}] / [2 (c_i + d_i s_i)]$
5. $\underline{x}_{i+1} = \underline{x}_i + frac*\gamma_i \underline{z}_i$ where frac is a suitable rate control parameter, controlled from the main loop
6. $\underline{x}_{i+1} = \underline{x}_{i+1} / || \underline{x}_{i+1} ||$ and $\lambda_{i+1} = \underline{x}_{i+1}^T R \underline{x}_{i+1}$

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