# THE DISCRETE-TIME FREQUENCY WARPED WAVELET TRANSFORMS

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## ABSTRACT

In this paper we show that the dyadic wavelet transform may be generalized to include non-octave spaced frequency resolution. We introduce orthogonal and complete wavelets whose set of cutoff frequencies may be adapted, in the simplest case, by changing a single parameter. The novel wavelets and the FWWT transform computational structure are obtained via an intermediate Laguerre representation of the signal. The warped wavelets are related to the ordinary wavelets by means of frequency transformations and orthogonalizing filtering. The classical sampled filter bank theory is extended to include frequency dependent upsampling and downsampling operators and dispersive delay lines. The FWWT frequency band flexibility may be exploited in order to adapt the wavelet transform to signals.

## 1. INTRODUCTION

The wavelet transform [4] is a tool for the multiresolution analysis and synthesis of signals. Dyadic wavelet sets allow for the efficient numeric computation of the transform by means of two-band iterated multirate filter banks. Efforts in the direction of increasing the frequency resolution of the basis functions or sequences rest prominently on the introduction of rational sampling rates [2, 10] and wavelet packets. These rates determine strict constraints on the passbands, conditioning the tiling of the time-frequency plane that is obtained. Furthermore, the regularity and smoothness of the wavelets may be severely impaired on sampling grids other than the dyadic one.

In this paper we exploit complete Laguerre expansions in order to both define a new class of wavelet bases, which have non-octave band frequency resolution, and novel computational structures for the transform. These wavelets may be obtained by frequency warping and suitably filtering the dyadic

wavelets. We refer to the resulting transform as the Frequency Warped Wavelet Transform (FWWT). A related result was presented in [1] for the continuous-time case. However, our wavelets can be implemented using an iterative scheme, which is computationally more attractive.

## 2. LAGUERRE EXPANSIONS AND FREQUENCY WARPING

Starting point of the FWWT algorithm is the expansion of a discrete time causal signal $y(k)$ on the complete orthonormal set of Laguerre sequences.

The Z-transform of the order $r$ sequence $\lambda_r(k;b)$, [3], is

$$\Lambda_r(z) = \sqrt{1-b^2}\,\frac{(z^{-1}-b)^r}{(1-bz^{-1})^{r+1}} \qquad . \qquad (1)$$

The Laguerre sequences satisfy the simple Z-transform recurrence:

$$\Lambda_{r+1}(z) = A(z)\Lambda_r(z) = A(z)^{r+1}\Lambda_0(z), \quad r=0,1,\dots, \quad (2)$$

where

$$A(z) = \frac{z^{-1}-b}{1-bz^{-1}}$$

is a first order all-pass, where $b$, with $|b|<1$, is a parameter. The Laguerre expansion of a signal $y(k)$ is the following

$$y(k) = \sum_{r=0}^{\infty} u_r \lambda_r(k;b), \qquad (3)$$

where

$$u_r = \sum_{k=0}^{\infty} y(k)\,\lambda_r(k;b). \qquad (4)$$

It can be shown that the coefficient sequence $u_r$ corresponds to a filtered frequency-warped version of the signal $y(k)$, [7, 9]. In fact, in the frequency domain we have:

$$Y(e^{j\omega}) = \Lambda_0(e^{j\omega})U(e^{j\theta(\omega)}), \qquad (5)$$

where $\theta(\omega)$ is the phase response of the all-pass $A(z)$. In the relevant case where $b$ is real, we have:

$$\theta(\omega) = - \text{arg } A(e^{j\omega}) = \omega + 2\tan^{-1}\left(\frac{b \sin\omega}{1 - b\cos\omega}\right).$$

$$(6)$$

These characteristics are plotted in fig. 1 for several values of $b$. Notice that this parameter controls the warping of the frequency axis. The filter

$$\Lambda_0(z) = \frac{\sqrt{1-b^2}}{1-bz^{-1}}, \qquad (7)$$

which is needed for orthogonality, is low-pass or high-pass according to the sign of the parameter.
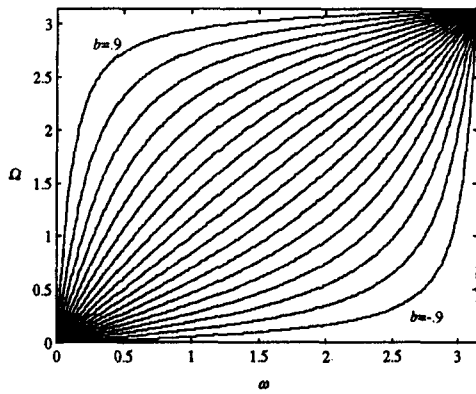


Fig. 1: Family of frequency warping curves with the Laguerre parameter b ranging from -.9 to .9 in increments of .1.

In view of eqs. (2, 3), the Laguerre transform may be implemented in the non-causal IIR filter structure shown in fig. 2.
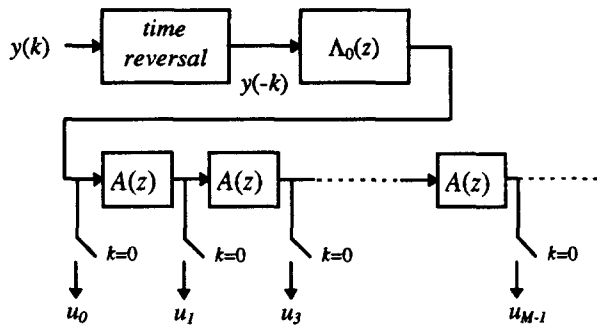


Fig. 2: A filter structure computing the Laguerre coefficients.

Clearly, only the transform of finite duration signals can be computed in finite time. Furthermore, for computational purposes, the Laguerre series must be truncated to a finite number $M$ of terms. In order to select the proper number of terms, one can consider the group delay of the Laguerre sequences, which is frequency dependent. Given a length $D$ input signal, the terms whose minimum group delay is much larger than $D$ may be neglected. A lower bound on the number $M$ of Laguerre coefficients that must be computed is the following :

$$M \geq M_{\min} \equiv \frac{(D-1)(1+|b|)+1}{1-|b|} \approx \frac{D(1+|b|)}{1-|b|}. \qquad (8)$$

Notice that the number of terms increases with $|b|$.

## 3. FREQUENCY WARPED WAVELETS

The sequence of Laguerre coefficients $u_r$ may be projected onto a set of discrete-time dyadic wavelets $\hat{\psi}_{n,k}(m)$. In its simplest form, the nested orthogonal expansion may be compactly written as follows :

$$y(k) = \sum_m \sum_{n=1}^{\infty} w_{n,m}\psi_{n,m}(k) ,$$

where

$$w_{n,m} = \sum_{k=0}^{\infty} y(k) \ \psi_{n,m}(k)$$

is the FWWT of the signal, and

$$\psi_{n,k}(m) = \sum_{r=0}^{\infty} \hat{\psi}_{n,k}(r)\lambda_r(m)$$

are the Laguerre wavelets, which are easily shown to be both orthogonal and complete [7]. The equivalent computational structure of the FWWT is shown in fig. 3.
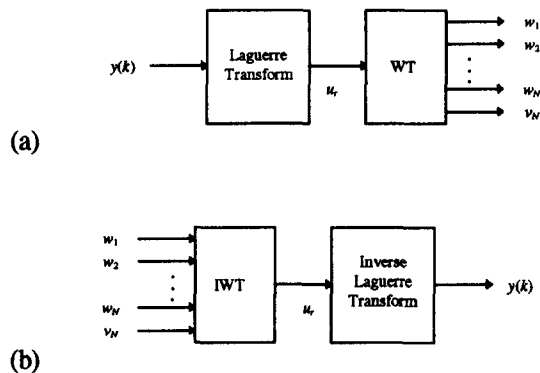


(a)

(b)

Fig. 3: Structure for computing the Laguerre wavelet transform: implementation with Laguerre and wavelet transform blocks: (a) analysis structure and (b) synthesis structure.

It can also be shown that these wavelets are frequency warped and filtered versions of the ordinary wavelets. Indeed, their Z-transforms satisfy the following relationship :

$$\Psi_{n,k}(z) = \Lambda_0(z)\hat{\Psi}_{n,k}(A(z)^{-1}) = \Lambda_{2^n k}(z)\hat{\Psi}_{n,0}(A(z)^{-1}),$$
$$(9)$$

which is closely related to eq. (5). Typical Laguerre wavelets are shown in fig. 4. The discrete-time FWWT may be extended to continuous-time by means of Laguerre interpolation [7].
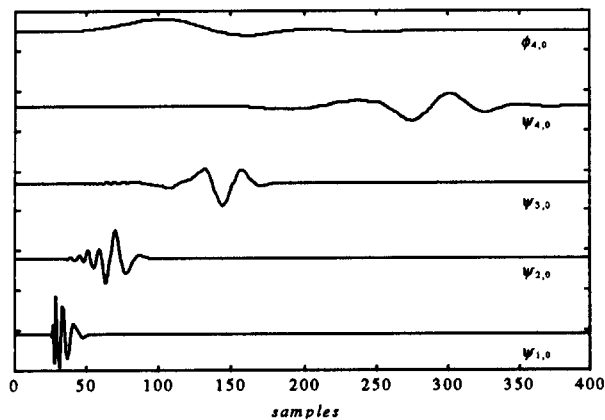


(a)
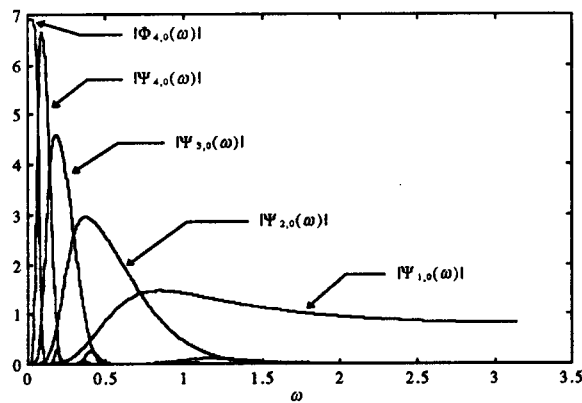


(b)

Fig. 4: (a) Typical Laguerre wavelets and (b) their frequency spectra corresponding b=.5.

By exploiting the Z-transform relationship among the Laguerre sequences (2) it is possible to derive from (9) new filterbank structures, embedding frequency warping in critically sampled filterbanks. These structures, shown in fig. 5, are based on generalized down/up sampling operators, whose effective sampling rates are frequency dependent, denoted by the symbols $A(z)^2 \downarrow$ and $A(z)^2 \uparrow$ respectively. The generalized downsampling operator is equivalent to a long chain of cascaded all-pass filters, i.e., a dispersive delay line similar to the one employed for the Laguerre expansion (see fig. 2), except that each all-pass element is the square of a first-order all-pass [7].
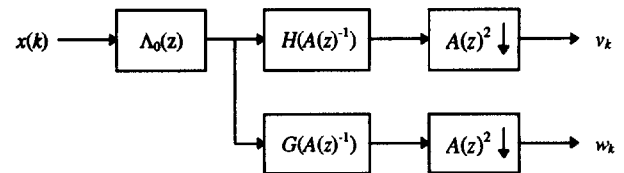


(a)
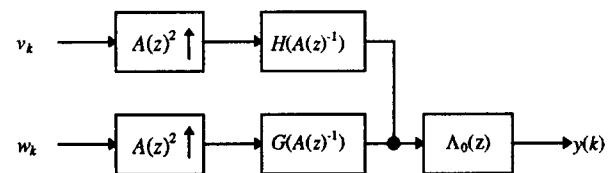


(b)

Fig. 5: Two-channel frequency warped filterbanks: (a) analysis structure and (b) synthesis structure.

The generalized upsampling operator is equivalent to a Laguerre filter structure [8], obtained from an FIR tapped delay line by replacing each delay with $A(z)^2$. The input to the line is a unit pulse, while the signal forms the tap weights. Notice that when the Laguerre parameter goes to zero the described all-pass delay line are equivalent to ordinary down/up sampling.

The Laguerre wavelets realize an unconventional tiling of the time-frequency plane in non-rectangular cells. An example is shown in fig. 6, where the central time of each tile is computed according to the group delay of the corresponding warped wavelet.

Similarly to the Laguerre expansion, the FWWT filter structures are non-causal IIR, requiring finite duration input sequences in order to be computed in finite time. Furthermore, the Laguerre series must be truncated to a finite number of terms, as previously discussed.

## 4. CONCLUSION

The increased effort needed to compute the FWWT, essentially due to the Laguerre transform block,

may be rewarding: experiments with this new transform show a large variety of new perspectives both from a conceptual and application viewpoint, removing the half-band constraints of ordinary wavelet filters. Frequency warped wavelets may be extended to pitch-synchronous wavelets [5, 6] to obtain an efficient representation of pseudo-periodic and multi-periodic signals with non uniformly spaced "harmonics" or partials. Furthermore, a simple generalization of the transform includes scale-dependent frequency warping, thus increasing the flexibility of the design and allowing for higher resolution in both low and high frequencies.
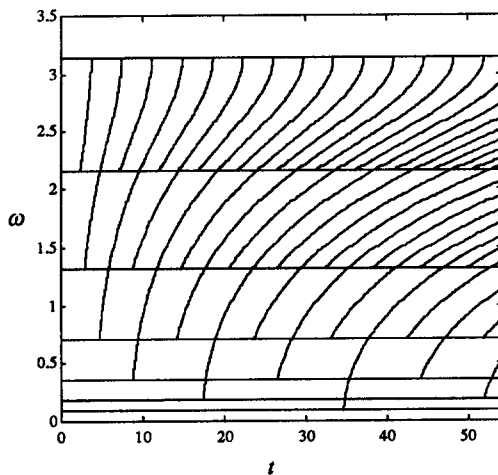


Fig. 6: Tiling the time-frequency plane with frequency warped wavelets with Laguerre parameter b=-.3.

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