

# BOUNDARY-COMPENSATED WAVELET BASES

*Mark A. Coffey*

Department of Electrical and Computer Engineering  
University of Colorado at Boulder, USA  
*Mark.A.Coffey@Boulder.Colorado.EDU*

## ABSTRACT

We investigate the formulation of boundary compensated wavelet transforms supported on a finite interval. A unified approach to boundary compensated wavelet transforms is presented which fosters new insights into previous constructions, including both continuous and discrete approaches to the problem. The framework enables the design of boundary-compensated transforms with specific properties, including among others arbitrary frequency response, matching moments, and staggered supports.

## 1. INTRODUCTION

To date, there remain unresolved issues concerning the application of wavelet transforms to finite data sets. Such issues as boundary conditions, sequence lengths, and irregular support regions prevent the realization of the full potential which wavelet analysis may have to offer. While for some applications these issues are easily resolved, there are others for which they become critically important, especially in areas such as geophysics or image processing where the data at the edge of a field is as, if not more, important than the interior. In our work with satellite altimetry data sets, we have realized the need for more general constructions of wavelet transforms which focus on eliminating boundary distortion effects. This proposal outlines a unified approach to constructing wavelet bases which live on the interval. This framework encompasses previous constructions and suggests variations which may better suit some applications, including control over moment properties and temporal response of the resulting edge functions. We have used the resulting constructions on the finitely-supported fields resulting from oceanographic surveys using satellite remote-sensing data.

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## 2. BOUNDARY-COMPENSATED WAVELET TRANSFORMS

The problem of edge effects in filtering is by no means a new one, although it affects the DWT (and other dyadic tree filter banks) in a unique fashion. There are several conventional methods of correcting for (or minimizing the effects of) boundary distortion in filters, including zero-padding, data reflection, windowing, and periodization, to name a few [1]. Although some show improvement, it is evident that none provide an ideal solution to the problem. All suffer from some degree of distortion. To make matters worse, any distortion introduced by filtering is made even worse by subsequent filtering inherent in the binary tree-structure used to implement wavelet transforms. These conventional means of treating the boundaries of data in signal processing are unsatisfactory, in that they address only the implementation part of the problem. They are used only in the algorithm used to implement the DWT. Other solutions to the boundary distortion problem exist if we examine instead the wavelet transform itself.

### 2.1. Wavelets on the interval: a unified framework

Although the DWT is implemented in discrete fashion, it enjoys a rich interpretation in the continuous domain in terms of the underlying basis scaling functions and wavelets. On the infinite line, we are guaranteed a wavelet representation for any function in  $L_2(\mathcal{R})$ . However, a compactly-supported function on an interval  $[a, b]$  does not necessarily have a representation in the scaling function basis which spans the same support of the function. In addition to the basis functions which have the same support as the function itself, we have several basis functions which will have some support both in and out of the function's interval of support. It is precisely the fact that we require basis functions whose support falls outside the interval of in-

terest which leads to the boundary distortion effects. A logical solution to the problem addresses the spillover of these edge functions into the region of support outside the function's support. Such constructions have been recently considered, and are generally known as wavelets on the interval. They are approached from two distinct, but related, viewpoints: continuous [2] and discrete constructions [3, 4]. We outline a unified framework from which to approach both constructions, including insights into their similarities and differences. This general framework can be shown to include several previous constructions found in the literature, as well as suggesting alternate constructions.

The starting point for the general construction is a standard biorthogonal wavelet basis with compact support [5], defined by the lowpass filter  $h$  and associated dual filter  $g$ . For a given filter length  $N$ , the associated basis functions will have support on  $[-\lfloor \frac{(N-1)}{2} \rfloor, \lceil \frac{(N-1)}{2} \rceil]$ . We wish to form boundary functions which are supported only on the half-infinite interval  $[0, \infty)$  (the right side of the interval is constructed in an analogous manner.) Since the resulting functions must live in the same multiresolution analysis (MRA) defined by the interior functions, we begin by defining  $\Phi_b(x)$ , a vector of  $r$  boundary scaling functions, via a dilation equation:

$$\Phi_b(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_{r-1}(x) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1^\phi & \mathbf{H}_2^\phi \end{bmatrix} \begin{bmatrix} \Phi_b(2x) \\ \Phi_2(2x) \end{bmatrix} \quad (1)$$

where  $\Phi_2(x)$  is the vector of untruncated interior scaling functions and  $\mathbf{H}^\phi = [\mathbf{H}_1^\phi \mathbf{H}_2^\phi]$  is the unknown dilation matrix. Similarly, define a vector of  $r$  boundary wavelets as:

$$\Psi_b(x) = \begin{bmatrix} \mathbf{H}_1^\psi & \mathbf{H}_2^\psi \end{bmatrix} \begin{bmatrix} \Phi_b(2x) \\ \Phi_2(2x) \end{bmatrix} \quad (2)$$

and the dual boundary basis:

$$\tilde{\Phi}_b(x) = \begin{bmatrix} \mathbf{G}_1^\phi \\ \mathbf{G}_2^\phi \end{bmatrix}^T \begin{bmatrix} \tilde{\Phi}_b(2x) \\ \tilde{\Phi}_2(2x) \end{bmatrix} \quad (3)$$

$$\tilde{\Psi}_b(x) = \begin{bmatrix} \mathbf{G}_1^\psi \\ \mathbf{G}_2^\psi \end{bmatrix}^T \begin{bmatrix} \tilde{\Phi}_b(2x) \\ \tilde{\Phi}_2(2x) \end{bmatrix} \quad (4)$$

Define the dilation operator as the matrix  $\mathbf{D}$  which dilates vectors by a factor of two:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \vdots \\ & & & & 1 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

and note that  $\mathbf{AD}$  dilates the rows of  $\mathbf{A}$ ,  $\mathbf{D}^T \mathbf{A}$  dilates the columns of  $\mathbf{A}$ , and  $\mathbf{DD}^T = \mathbf{I}$ . The vector of all boundary functions is then given as

$$\Theta_b(x) = \begin{bmatrix} \mathbf{D}^T \Phi_b(x) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^T \Psi_b(x) \end{bmatrix} \quad (6)$$

The problem then is the determination of appropriate matrices  $\mathbf{H}$  and  $\mathbf{G}$  such that the underlying boundary functions maintain the structure of the interior MRA as closely as possible. The boundary filters defined by the rows of the dilation matrices can then be used in conjunction with the standard interior filters to form a boundary-compensated wavelet transform.

The boundary functions associated with any choice for  $\mathbf{H}$  and  $\mathbf{G}$  must satisfy biorthogonality conditions. The grammian associated with the boundary functions is

$$\mathbf{S}_{\Theta_b, \tilde{\Theta}_b} = \int_0^\infty \Theta_b(x) \tilde{\Theta}_b(x)^T dx \quad (7)$$

and can be explicitly evaluated using

$$\mathbf{S}_{\Theta_b, \tilde{\Theta}_b} = \mathbf{H}_1 \mathbf{D} \mathbf{S}_{\Phi_b, \tilde{\Phi}_b} \mathbf{D}^T \mathbf{G}_1 + \mathbf{H}_2 \mathbf{G}_2 \quad (8)$$

Biorthogonality requires that  $\mathbf{S}_{\Theta_b, \tilde{\Theta}_b} = \mathbf{I}$ , a condition sufficiently satisfied when  $\mathbf{HG} = \mathbf{I}$ . In addition, the new boundary functions must remain biorthogonal to the original interior functions.

Biorthogonality alone does not guarantee boundary functions which will satisfy the underlying MRA, however. In addition, there are standard admissibility conditions which must be met. Although stronger than necessary, the conditions  $\int_0^\infty \phi_n(x) dx = 1$  and  $\int_0^\infty \psi_n(x) dx = 0$  suffice. In order to evaluate these expressions, we require the ability to calculate moments of the boundary functions. Define

$$M_{l,k}^\phi \equiv \int_0^\infty x^l \phi(x - k) dx \quad (9)$$

the  $l^{th}$  moment of  $\phi$  shifted by  $k$ . (When omitted, the parameter  $k$  is assumed to be zero.) Then the  $l^{th}$  moment of the boundary functions is calculated as:

$$\mathbf{M}_l^{\Phi_b} = \left[ 2^{(l+\frac{1}{2})} \mathbf{I} - \mathbf{H}_1^\phi \right]^{-1} \mathbf{H}_2^\phi \mathbf{M}_l^{\Phi_2} \quad (10)$$

The moments of the interior functions are known explicitly from the interior filter coefficients. In order to satisfy  $\int_0^\infty \phi_n(x) dx = 1$ , we can solve (10) for  $\mathbf{M}_l^{\Phi_b} = \mathbf{1}$ . Since the interior scaling functions integrate to one by definition, this yields the condition  $\mathbf{H}\mathbf{1} = \mathbf{1}$ , or that the boundary filter coefficients must sum to unity, just like the interior coefficients. Similar arguments can be used to show that the boundary wavelet filters must sum to zero.

These conditions guarantee the existence of underlying basis functions which are admissible. However, the resulting bases can be further constrained so as to help minimize the distortion yielded by the boundary basis. The boundary basis functions can be constructed in such a manner that their low-order moments will agree with the moments of the truncated interior basis functions which they are replacing. As above, this corresponds to restrictions on weighted sums of the boundary filter coefficients. The result is a transform which is less sensitive to edge effects.

Because of the intimate ties between the DWT matrices and the underlying continuous basis functions, it is relatively easy to impose other characteristics on the continuous wavelet bases produced. QR factorization can be used to provide boundary functions with staggered supports. The family of orthogonal bases is also included in this framework; indeed, a few applications of the SVD algorithm yields orthonormalized versions of the bases produced here [6]. Depending on the application for which the basis is being constructed, there may be criterion more important to consider than the ones presented here. This could include equal-length filters, coiflet-like boundary functions, nearly-orthogonal functions, and/or filters with a specific phase response.

## 2.2. Construction of the boundary functions

A useful advantage to viewing the boundary problem using the notation of the previous section is that it leads to relatively simple algorithms by which the actual boundary filters may be constructed. Simply put, the design problem is to construct two finite-dimensional matrices which are mutually inverse. Begin with  $\mathbf{H}_i$ , the truncated DWT convolution matrix associated with the interior filters, and choose candidate boundary filters as the  $r$  truncated filters whose support falls outside the left boundary. Define  $\mathbf{G}_i$  in a similar fashion. Project the candidate filters onto the nullspace of  $\mathbf{G}_i$ , yielding filters orthogonal to the columns of  $\mathbf{G}_i$ . These filters  $\mathbf{h}$  and any linear combination of them  $\mathbf{A}\mathbf{h}$  will remain in the nullspace of  $\mathbf{G}_i$ . We then wish an appropriate combination of filters with desirable moments properties. In particular we require

$$\mathbf{A}\mathbf{h} \begin{bmatrix} \mathbf{v}^0 & -\mathbf{v}^0 & \mathbf{v}^1 & \dots & \mathbf{v}^l \end{bmatrix} = \begin{bmatrix} \mathbf{1}_r \otimes \mathbf{I}_2 & \mathbf{M}_1^{\Phi_1} & \dots & \mathbf{M}_l^{\Phi_1} \end{bmatrix} \quad (11)$$

where  $\mathbf{v} = [0 \ 1 \dots (N-1)]^T$  and  $-\mathbf{v}$  is the modulated version of  $\mathbf{v}$ . There are additional degrees of freedom in (11) for a basis of given vanishing moments that can be exploited. We use these to influence the temporal impulse response of the filters. In particular, we would

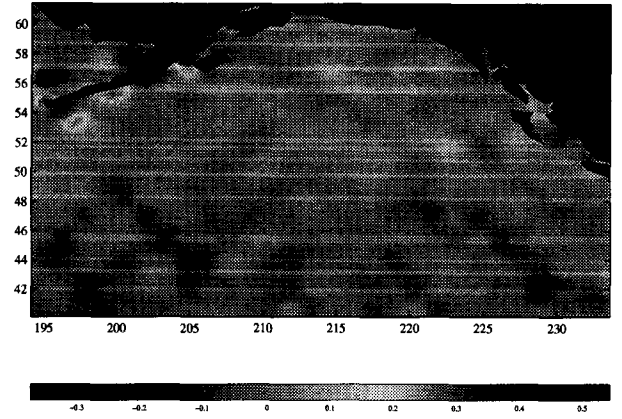


Figure 1: SSH field in the Gulf of Alaska as viewed by the TOPEX altimeter.

like the boundary filters to resemble their truncated counterparts as closely as possible. This is equivalent to solving

$$\min \|\mathbf{A} - \mathbf{I}\|^2 \text{ s.t. } \|\mathbf{A}\mathbf{X} - \mathbf{Y}\|^2 = 0 \quad (12)$$

Vectorizing the equations and applying Lagrange multipliers yields the solution for  $\mathbf{A}$ . Once the analysis filters are determined, the dual filters are then calculated by projecting truncated dual filters onto the row nullspace of  $\mathbf{H}_i$  and normalizing. The resulting  $\mathbf{H}$  and  $\mathbf{G}$  are then mutually inverse at the left boundary and interior. Repeating the above procedure at the right boundary completes the construction.

## 3. APPLICATION

Using the method outlined in Section 2.2, we constructed a biorthogonal basis which minimized the boundary distortion effects seen in applying the DWT to finite two-dimensional data sets. As an example of the effectiveness of the constructed basis, consider the image in Figure 1 showing the sea surface height (SSH) field in the Gulf of Alaska as measured by the TOPEX satellite altimeter. Taking the boundary compensated DWT of the image using the construction of wavelets on the interval in [2], one obtains the analysis shown below in Figure 2. The distortion introduced along the edges of the image are an order of magnitude larger than the interior magnitudes, making it difficult to discern trends in the interior, and completely masking trends at the image boundary.

The basis constructed in the previous section is used on the same image, with results shown in Figure 3. Because the basis was constructed so as to minimize boundary distortion effects, the analysis subspaces are

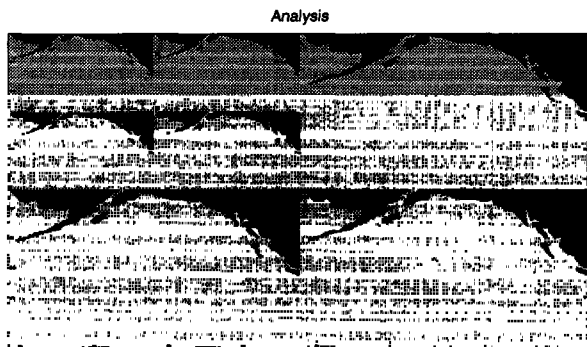


Figure 2: 2D-DWT of the image in Figure 1 based on CDV bases of order 3.

much cleaner, with trends tracking even at the image boundaries. This represents a clear advantage in terms of subjective scientific image analysis using wavelet transforms. The basis is also “nearly orthogonal”, enabling signal separation using partial subspace reconstruction with relatively little distortion introduced by aliasing. However, if partial reconstruction is a major criterion for designing a wavelet basis, then biorthogonal solutions may not be the best choice.

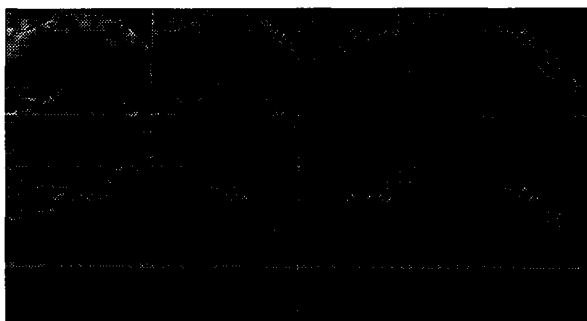


Figure 3: DWT based on constructed biorthogonal bases with three vanishing moments.

#### 4. CONCLUSION

The new framework has several advantages over previous construction methods. It provides a common notation and derivation which includes many of the constructions published to date. This affords a means by which one may compare constructions. A major advantage to the framework outlined here is that it is

vectorized. This allows the use of standard linear algebra to achieve desirable properties of the resulting boundary basis functions. For example, QR factorization can be performed, yielding a unitary transformations of one set of boundary functions into another set with staggered supports. A number of other results from unitary matrix theory can be used to achieve other properties for the boundary functions. Still other algorithms such as the SVD can be used to perform quick and efficient orthonormalization. Another benefit of the new construction is the existence of efficient algorithms for explicitly computing such quantities as the moments of the boundary functions. We have derived algorithms which can quickly compute boundary moments, shifted moments, continuous approximations, and others. These can be used to study regularity properties and continuity of the boundary functions at the interval edge.

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