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ABSTRACT

This paper studies the applicability and limitations of the McClellan transformation method and, as a result, extends this method so that new types of one-dimensional filters can be transformed and new types of multi-dimensional filters can be designed. For this purpose, a new expression for the frequency response of an arbitrary one-dimensional filter is derived in terms of Chebyshev polynomials and other introduced polynomials satisfying recurrence formulae. The main objective is to identify which prototype filters can be transformed, determine what types of symmetry can be designed, and present procedures for transforming the new identified prototypes as well as rules for achieving the possible symmetries.

1. INTRODUCTION

The design of optimal (equiripple) multi-dimensional FIR filters is much more involved than the one-dimensional design problem. Since the direct design and realization of optimal multi-D filters is expensive, transformations have been used to efficiently design and realize near-optimal multi-D filters.

A very efficient and popular transformation method is due to McClellan [1]. This technique transforms a 1-D prototype filter $H(\omega)$ into a 2-D filter $H(\omega_1, \omega_2)$ by a change of variables. The transformation is performed by using a transformation function $F(\omega_1, \omega_2)$ which would map the prototype filter to the desired 2-D filter. The McClellan transformation method allows the designer to control the error norm and to preserve the symmetry properties of the desired 2-D filters. In practice, it results in very good filters that are close to optimal. However, in the original McClellan transformation method, the prototype filter $H(\omega)$, the transformation function $F(\omega_1, \omega_2)$, and the 2-D designed filter $H(\omega_1, \omega_2)$ are restricted to be real and positive-symmetric [1]. This paper studies the applicability and limitations of the McClellan transformation method and, as a result, extends this method so that new types of one-dimensional filters can be transformed and new types of multi-dimensional filters can be designed. Design procedures are presented for the design of complex and real, positive- and negative-symmetric, multi-D filters by transforming odd- and even-length one-D prototype filters with complex (or real) coefficients.

2. IDENTIFICATION OF TRANSFORMABLE PROTOTYPE FILTERS

This section identifies the class of prototype filters that can be mapped into multi-dimensional filters by means of a transformation function. The discussion emphasizes the use of "well-behaved" transformation functions, i.e., functions that do not distort but exactly map the values of the prototype frequency response to corresponding contours in the multi-dimensional frequency domain. Well-behaved mappings are important because they allow the designer to control the values of the designed multi-dimensional filter and, therefore, control the approximation error norm.

A general complex-valued one-dimensional FIR filter of length N , with the coefficients $\{h_k; k = K_1, \dots, K_1 + N - 1\}$, has a frequency response $H(\omega)$ given by

$$H(\omega) = \sum_{k=K_1}^{K_1+N-1} h_k e^{-jk\omega} \quad (1)$$

Alternatively, it is shown that $H(\omega)$ can be expressed in terms of Chebyshev and other related polynomials satisfying recurrence formulae. In fact, rewriting $e^{-j\omega k}$ as $\cos(k\omega) - j \sin(k\omega)$, the functions $\cos(k\omega)$ and $\sin(k\omega)$ can be expressed as

$$\cos(k\omega) = T_k[\cos \omega] \quad (2)$$

and

$$\sin(k\omega) = \begin{cases} S_k[\sin \omega], & k \text{ odd} \\ S_k[\sin \omega] \cos \omega, & k \text{ even} \end{cases} \quad (3)$$

where $T_k[x]$ is the well-known k th Chebyshev polynomial of degree k in x and $S_k[x]$ is an introduced polynomial which is shown to satisfy the following recurrence formula:

$$\begin{aligned} S_k[x] &= \begin{cases} 2S_{k-1}[x](1-x^2) - S_{k-2}[x], & k \text{ odd} \\ 2S_{k-1}[x] - S_{k-2}[x], & k \text{ even} \end{cases} \\ S_0[x] &= 0 \\ S_1[x] &= x \end{aligned} \quad (4)$$

Note that $S_k[x]$ is a polynomial whose degree is k for k odd, and $k-1$ for k even. $S_k[x]$ is also shown to be related to $T_k[x]$ as follows:

$$S_k[x] = \begin{cases} (-1)^m T_{2m+1}[x], & k = 2m + 1 \\ 2(-1)^{m-1} \sum_{p=1}^m T_{2p-1}[x], & k = 2m \neq 0 \end{cases} \quad (5)$$

Now, using (2), $H(\omega)$ can be rewritten as

$$H(\omega) = \sum_{k=K_1}^{K_1+N-1} c_k T_{|k|}[\cos \omega] + \sum_{\substack{k=K_1 \\ k \text{ odd}}}^{K_1+N-1} \left(\frac{k}{|k|}\right) s_k S_{|k|}[\sin \omega] \\ + \sum_{\substack{k=K_1 \\ k \text{ even}}}^{K_1+N-1} \left(\frac{k}{|k|}\right) s_k S_{|k|}[\sin \omega] \cos \omega. \quad (6)$$

Equation (6) indicates that two real transformation functions are needed to transform an arbitrary filter by means of the two substitutions

$$\begin{cases} \cos \omega & \longleftrightarrow F_1(\omega_1, \omega_2) \\ \sin \omega & \longleftrightarrow F_2(\omega_1, \omega_2) \end{cases} \quad (7)$$

where the mapping functions $F_1(\omega_1, \omega_2)$ and $F_2(\omega_1, \omega_2)$ must be the frequency responses of 2-D FIR filters with $-1 \leq F_1(\omega_1, \omega_2) \leq 1$ and $-1 \leq F_2(\omega_1, \omega_2) \leq 1$. Furthermore, in order to preserve the characteristics of the one-dimensional prototype $H(\omega)$, $F_1(\omega_1, \omega_2)$ and $F_2(\omega_1, \omega_2)$ must satisfy the following additional constraint

$$F_1^2(\omega_1, \omega_2) + F_2^2(\omega_1, \omega_2) = 1. \quad (8)$$

Although arbitrary prototype filters (with no symmetries) can be transformed, it is shown that the required constraints on the mapping functions limit the choice of admissible mappings and, therefore, the filters that can be designed to only those with colinear isopotentials. In this case, a broader selection of transformation mappings with arbitrary contours could be obtained if the constraint (8) is not satisfied exactly but is satisfied in an approximative sense as described briefly below.

Transforming Arbitrary Prototype Filters The following procedure can be used for transforming an arbitrary prototype (with no exploitable symmetries). The substitution (7) corresponds to a transformation of variables which maps the 1-D frequency ω to a contour in the 2-D frequency space (ω_1, ω_2) . Let the desired contour mapping be

$$\omega = C_d(\omega_1, \omega_2) \quad (9)$$

where $C_d(\omega_1, \omega_2)$ is a function representing the desired contour in the 2-D space. Then, the mapping functions $F_1(\omega_1, \omega_2)$ and $F_2(\omega_1, \omega_2)$ of (7) could be constructed such that they best approximate $\cos(C_d(\omega_1, \omega_2))$ and $\sin(C_d(\omega_1, \omega_2))$, respectively, relative to some error norm [2]. It should be noted that $F_1(\omega_1, \omega_2)$ and $F_2(\omega_1, \omega_2)$ are real-valued FIR frequency responses. Note that, in this case, the values of the prototype frequency response may be altered by the mapping process. This is due to the fact that the constraint (8) is not exactly satisfied.

Exactly Transformable Prototype Filters Using (6), it is shown that well-behaved mapping functions, which do not alter but exactly map values of the prototype frequency response to *general* (arbitrary shape) contours in the multi-D frequency domain, can be used only with a limited class of prototype filters satisfying some symmetry conditions. Well-behaved transformation functions are important because they allow the designer to easily control the values

of the designed filter and to control the error norm. The well-behaved transformation functions are real-valued resulting in multi-dimensional filters whose rectangular regions of support have odd-length sides. Even-length prototype filters can be transformed, in some special cases, to produce 2-D filters with a rectangular region of support having even-length sides but with very restricted (not arbitrary) contours [3].

The required symmetry conditions on the prototype filter are derived for both odd- and even-length, complex and real prototype filters using expression (6) for the prototype frequency response. From the previous discussion, it can be easily seen that well-behaved mappings, approximating arbitrary contours, can only be used in the cases where (6) reduces to an expression in terms of only one of the trigonometric functions $\cos()$ and $\sin()$. The reduction of (6) is done by considering all possible combinations of the three summation terms and using the derived relation (5). The corresponding prototype can then be mapped into a multi-D filter by replacing the trigonometric function ($\cos()$ or $\sin()$) by any real mapping function F satisfying $|F| \leq 1$. The *exactly* transformable odd-length prototypes, with no restrictions placed on the shape of the generated contours, were found to be of the form

$$H(\omega) = \sum_{n=0}^{(N-1)/2} c_n T_n[x(\omega)] \quad (10)$$

where $x(\omega) = \cos(\omega)$ or $\sin(\omega)$. Similarly, the *exactly* transformable even-length filters were found to be of the form

$$H(\omega) = e^{-j\omega/2} x(\omega) \sum_{n=0}^{(N/2)-1} c_n T_n[s(2(x(\omega))^2 - 1)] \quad (11)$$

where

$$\begin{cases} x(\omega) = \cos(\omega/2) & \text{and } s = 1 & (> 0 \text{ sym.}) \\ \text{or} \\ x(\omega) = \sin(\omega/2) & \text{and } s = -1 & (< 0 \text{ sym.}) \end{cases} \quad (12)$$

In this case, the design procedure can be simply described as follows. The transformed multi-D filter is obtained by replacing $x(\omega)$ in (10) or (11) by a mapping function F that approximates the contours of the desired multi-D filter. Note that the coefficients c_n in (10) and (11) are not restricted to be real but are complex in general.

3. IDENTIFICATION OF ACHIEVABLE MULTI-D SYMMETRIES

This section considers all possible centro and quadrant symmetries that can be achieved using the transformation method and derives rules for selecting the appropriate pair of prototype filter and transformation mapping that would result in the desired 2-D symmetries. The exactly transformable prototype filters (10) and (11) are considered for this purpose. Generalization to higher dimensions can be easily obtained by increasing the dimensionality of the transformation function.

For 2-D complex FIR filters whose impulse response $h(n_1, n_2)$ is non-zero over a rectangular region of support

having odd-length sides, only the following four types of centro and quadrant symmetries are possible in general:

Positive centrosymmetry: $h(n_1, n_2) = h(-n_1, -n_2)$ For any prototype filter of the form (10) or (11), a sufficient condition for obtaining a positive centro-symmetric 2-D filter $H(\omega_1, \omega_2)$ is to have a positive centro-symmetric mapping $F(\omega_1, \omega_2) = F(-\omega_1, -\omega_2)$. Positive centro-symmetric 2-D filters can also be obtained by using mappings with negative symmetries

$$F(\omega_1, \omega_2) = -F(-\omega_1, -\omega_2) \quad (13)$$

or

$$\begin{cases} F(\omega_1, \omega_2) = \pm F(-\omega_1, -\omega_2) \\ F(-\omega_1, \omega_2) = \mp F(\omega_1, -\omega_2) \end{cases} \quad (14)$$

where $0 \leq \omega_1, \omega_2 \leq \pi$. However, in these cases, the prototype filter has to have an odd length and be positive symmetric with respect to both $\omega = 0$ and $\omega = \pi/2$. This latter requirement is needed to eliminate the odd-degree Chebyshev polynomials in (10).

Negative centrosymmetry: $h(n_1, n_2) = -h(-n_1, -n_2)$ From the discussion above, it can be deduced that $F(\omega_1, \omega_2)$ cannot be positive centro-symmetric in this case. For any prototype filter of the form (10) or (11), a negative centro-symmetric complex 2-D filter is obtained by using a negative centro-symmetric mapping $F(\omega_1, \omega_2) = -F(-\omega_1, -\omega_2)$. In addition, if an odd-length prototype filter is used, it needs to be positive symmetric with respect to $\omega = \pi/2$ ($\omega = 0$) and negative symmetric with respect to $\omega = 0$ ($\omega = \pi/2$). This latter requirement eliminates the even-degree Chebyshev polynomials in (10). No restrictions are placed on the even-length prototype filters.

Positive quadrant symmetry: $h(n_1, n_2) = h(-n_1, -n_2) = h(-n_1, n_2) = h(n_1, -n_2)$ This case corresponds to a positive centro-symmetric filter with an additional quadrant symmetry constraint. For any prototype filter of the form (10) or (11), a sufficient condition for designing a 2-D filter which is positive quadrant symmetric is to have a positive quadrant symmetric transformation function $F(\omega_1, \omega_2) = F(-\omega_1, -\omega_2) = F(-\omega_1, \omega_2) = F(\omega_1, -\omega_2)$. With an odd-length filter (10) having $c_{2m+1} = 0$, other types of symmetries are possible for $F(\omega_1, \omega_2)$, i.e.,

$$\begin{aligned} F(\omega_1, \omega_2) &= s_1 F(-\omega_1, \omega_2) = s_2 F(-\omega_1, -\omega_2) \\ &= s_3 F(\omega_1, \omega_2) \end{aligned} \quad (15)$$

where s_1, s_2 , and s_3 are either -1 or 1 and are not necessarily equal.

Negative quadrant symmetries In this case, the following symmetry properties are considered for the 2-D filter $H(\omega_1, \omega_2)$:

$$\begin{aligned} H(\omega_1, \omega_2) &= s_1 H(-\omega_1, \omega_2) = s_2 H(-\omega_1, -\omega_2) \\ &= s_3 H(\omega_1, \omega_2) \end{aligned} \quad (16)$$

where s_1, s_2 , and s_3 are either -1 or 1 with exactly two of them equal to -1 . Such 2-D symmetries are obtained by using a transformation function $F(\omega_1, \omega_2)$ with the same type of symmetry as the desired 2-D filter, i.e.,

$$\begin{aligned} F(\omega_1, \omega_2) &= s_1 F(-\omega_1, \omega_2) = s_2 F(-\omega_1, -\omega_2) \\ &= s_3 F(\omega_1, \omega_2) \end{aligned} \quad (17)$$

where s_1, s_2 , and s_3 are as in (16). In addition, if an odd-length prototype (10) is used, it needs to have zero even-indexed coefficients, i.e., $\{c_{2m}\} = 0$. No restrictions are placed on the even-length prototype filters.

4. DESIGN EXAMPLE

This example corresponds to designing a bandpass, negative centro-symmetric 2-D filter, whose passbands are centered on the $\omega_1 = -\omega_2$ line, by transforming an appropriate odd-length 1-D prototype. From the rules stated in the previous section, a negative centro-symmetric mapping $F(\omega_1, \omega_2)$ is needed and the odd-length 1-D prototype filter needs to be negative symmetric. So, a first-order negative-symmetric mapping of the general form

$$\begin{aligned} F(\omega_1, \omega_2) &= A \sin(\omega_1) + B \sin(\omega_2) + C \sin(\omega_1) \cos(\omega_2) \\ &\quad + D \cos(\omega_1) \sin(\omega_2) \end{aligned} \quad (18)$$

is used. The odd-length 1-D prototype filter is designed to approximate the following ideal specifications:

$$D(\omega) = \begin{cases} 1, & (\frac{\pi}{2} - \omega_p) < \omega < (\frac{\pi}{2} + \omega_p) \\ -1, & (-\frac{\pi}{2} - \omega_p) < \omega < (-\frac{\pi}{2} + \omega_p) \\ 0, & \text{otherwise} \end{cases} \quad (19)$$

where ω_p controls the passband cutoff frequency and is such that $0 < \omega_p < (\pi/2)$. Figure 1 shows the properties of the optimal prototype filter with a length $N = 31$ and $\omega_p = 0.1\pi$. This optimal filter was obtained using the multiple-exchange ascent algorithm of [4]. To determine the parameters of the first-order mapping $F(\omega_1, \omega_2)$, the following desired mapping constraints are applied:

$$(\omega, \omega) \longleftrightarrow 0 \quad (20)$$

$$(\omega, -\omega) \longleftrightarrow \omega \quad (21)$$

where (20) maps $\omega = 0$ to the line $\omega_1 = \omega_2$, and (21) maps the prototype filter to the line $\omega_2 = -\omega_1$. The resulting mapping function is given by

$$F(\omega_1, \omega_2) = 0.5(\sin \omega_1 - \sin \omega_2). \quad (22)$$

The contours of $F(\omega_1, \omega_2)$ are shown in Fig. 2, and the frequency response of the designed 2-D bandpass filter is shown in Fig. 3.

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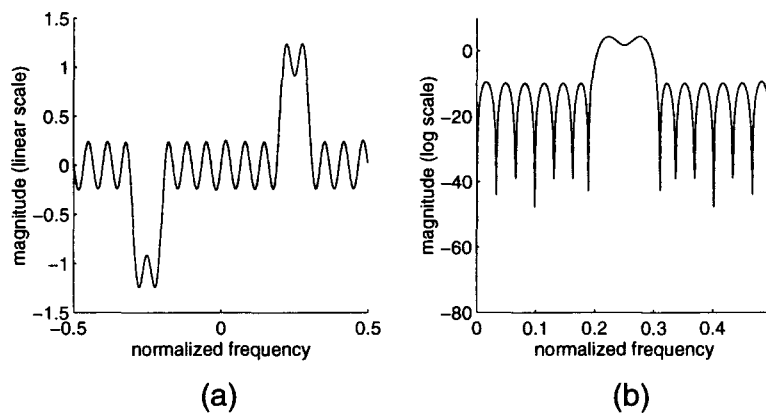


Figure 1. Prototype filter: $N = 31$ and $\omega_p = \pi/10$. (a) FIR filter magnitude response (linear scale). (b) FIR filter magnitude response (in dB).

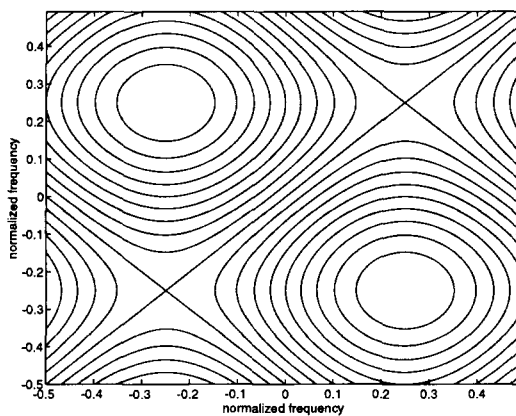


Figure 2. Contours of the negative centro-symmetric subfilter.

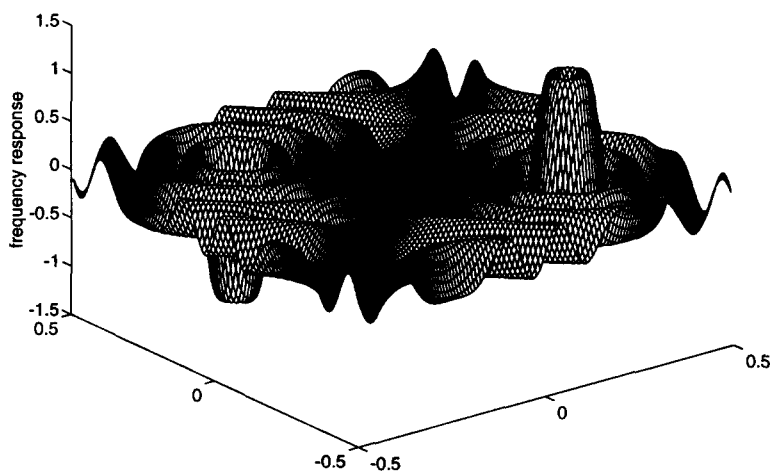


Figure 3. Frequency response of the designed 2-D filter, $\omega_p = \pi/10$.