DESIGN OF MULTIPLIERLESS ELLIPTIC IIR FILTERS

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ABSTRACT

A new straightforward design of a special class of elliptic IIR filters is presented. The major goal is the complexity reduction of the realized digital filter. The multiplier coefficients are implemented with limited number of shift-and-add operations. This method is also called "multiplierless". Unlike classical design, a closed-form relations are derived giving relationships between the filter specification and preferred multiplier coefficients. At least a half of the coefficients can be implemented with the minimal number of shift-and-add operations without coefficient quantization. The second half of coefficients can be optimized without any influence on the values of the first half of the coefficients. A high attenuation margins and low-sensitive structures are used so that specification is still fulfilled after quantization of the second half of the multiplier coefficients.

1. INTRODUCTION

The complexity of a digital filter, when implemented as a custom or semi-custom integrated circuit, a reprogrammable logic device or a low-cost micro-controller without an in-built multiplier, is determined primarily by the number of additions required to implement multiplication constants. This imposes a requirement to fulfil given specifications with a minimum number of shift- and add-operations in multipliers, what practically leads to the implementation of multiplierless filters. In papers [1] and [2] this problem has been solved for FIR filters. As for IIR filters, it has been shown in [3] that the use of different wordlengths for chosen coefficients, in accordance with different magnitude response sensitivities, can reduce the mean coefficient wordlength.

The aim of this paper is to introduce a direct design method for the elliptic IIR filters in which each multiplication constant can be represented in the form: $\pm 1/2^p$, or $\pm 1/2^p \pm 1/2^q$ or $\pm 1/2^p \pm 1/2^q \pm 1/2^r$, p, q, r integers, what practically yields a multiplierless implementation. This method can also be employed in designing very complex filters, because an elliptic function digital filter is an optimal solution in very selective magnitude nonlinear phase and linear phase IIR filters [4]. It is shown in the paper that with an adequate usage of the practically always existing margin in performance, a multiplierless filter can be obtained by a convenient choice of the filter transfer function and the realization structure matched with it.

The transfer function $H(z)$ is formed by a bilinear transformation from an analog minimal Q factors prototype [5]. The poles of $H(z)$ are in the z plane on a circle that is orthogonal to the unit circle and whose centre is on the real axis [6]. This halves the number of parameters required for representing the poles in the z plane: one parameter is common to all poles (the centre of the circle), and the position of each single pole is determined by its radius only. The centre of the circle depends exclusively on the frequency at which the filter has a 3 dB attenuation. The square magnitude response of this filter has equal pass- and stop-band tolerances ($\delta_p = \delta_a$) what gives a very small pass-band attenuation.

2. IMPLEMENTATION STRUCTURE

The implementation based on the sum of or difference between two allpass functions is used, i.e.

$$H(z) = \frac{1}{2}\ \left(H_b(e^{j\omega}) \pm H_a(e^{j\omega})\right) \qquad (1)$$

As known, this is the most economical implementation, because it requires a total of n multiplications, n - the filter order, an odd number. If $H(z)$ is a transfer function of an odd-order elliptic filter, it can be presented in the form:

$$H(z) = \frac{1}{2}\left(z \prod_{1}^{[(n+3)/4]} \frac{\beta_i + \alpha_i(1+\beta_i)z^{-1} + z^{-2}}{1 + \alpha_i(1+\beta_i)z^{-1} + \beta_i z^{-2}} \pm \prod_{[(n+7)/4]}^{(n+1)/2} \frac{\beta_i + \alpha_i(1+\beta_i)z^{-1} + z^{-2}}{1 + \alpha_i(1+\beta_i)z^{-1} + \beta_i z^{-2}} \right) \qquad (2)$$

If the position of the pole z_i is given by $z_i = r_i\ e^{\pm j\theta_i}$, α_i and β_i are determined from

$$\begin{cases} \beta_1 = 0 \\ \alpha_1 = -r_1 \end{cases} \quad \text{and} \quad \begin{cases} \beta_i = r_i^2 \\ \alpha_i = -2\dfrac{r_i \cos\theta_i}{1+r_i^2} \end{cases} \quad i > 1 \qquad (3)$$

what gives the parameters needed for the implementation of allpass second-order sections according to [7].

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3. DESIGN PROCEDURE

Let the required digital filter specifications be given with boundary frequencies for the pass-band F_p and F_a for the stop-band, pass-band ripple A_p and minimal stop-band attenuation A_a expressed in dB, as shown in Fig. 1. The filter specifications are fulfilled for various combinations of the elliptic filter boundary frequencies f_p and f_a. Design margin from Fig. 1 $[a_a - A_a, A_p - a_p, F_a - f_a, f_p - F_p]$ can be used for obtaining a minimal number of adders in multiplication constants α_i and β_i.

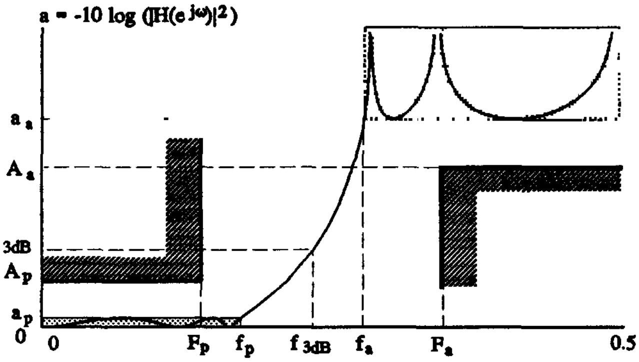

Figure 1. A typical elliptic filter

To achieve this, we will divide the multiplication constants into two groups that will be considered separately: the first group includes α_i and the second β_i. It is proved directly that, for the filter transfer function proposed in this paper, the constants from the first group are:

$$\alpha_i = \alpha, \qquad i > 1$$

$$\alpha_1 \approx \frac{\alpha}{2} + \frac{\alpha^3}{8} + \frac{\alpha^5}{16} \tag{4}$$

where

$$\alpha \approx -\frac{1 - \tan \pi F_p \tan \pi F_a}{1 + \tan \pi F_p \tan \pi F_a} \tag{5}$$

$$-\frac{1 - \tan^2 \pi F_p}{1 + \tan^2 \pi F_p} < \alpha < -\frac{1 - \tan^2 \pi F_a}{1 + \tan^2 \pi F_a} \tag{6}$$

The given F_p and F_a are used to determine the range of the permissible values of α,

$$\alpha = \cos 2\pi f_{3\text{dB}}, \tag{7}$$

$$\tan^2 \pi f_{3\text{dB}} = \tan \pi f_p \tan \pi f_a \tag{8}$$

The first step is to see if anyone of α from Fig. 2 belongs to the range defined in (6), and is also close to the approximate values (5). Similarly to Fig. 2, Figs. 3 and 4 are created presenting the values which can be made by the sum or difference of two and three coefficients, respectively. The next step is to see if from Figs. 3 and 4, a value α can be selected such as to lie in the range (6). This is how the frequency $f_{3\text{dB}}$ is established. This frequency must remain unchanged during the entire procedure, whereas f_a can be modified yet keeping relation (8) satisfied. The established value of α is common to all second-order sections and is repeated $(n-1)/2$ times.

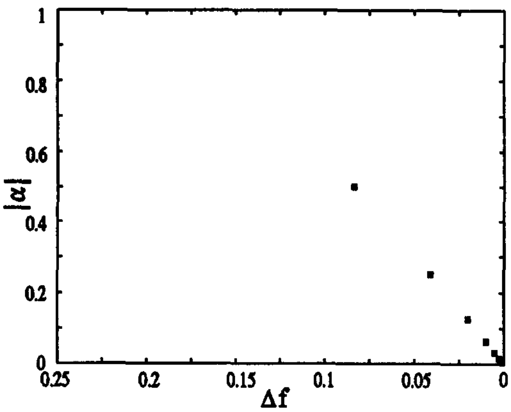

Figure 2. $f_{3dB} = 1/4 \pm \Delta f$, $\alpha \in \{\pm 1/2^p\}$, $p = 0, 1, \ldots 8$.

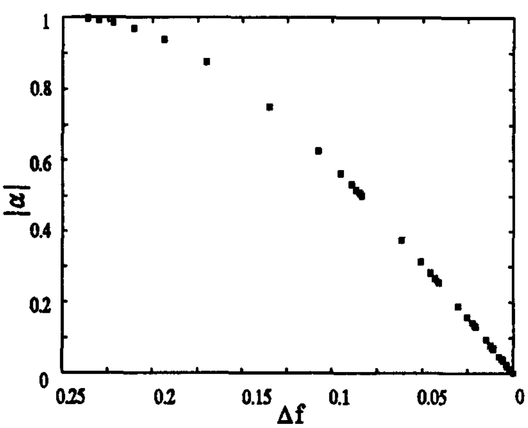

Figure 3. $f_{3dB} = 1/4 \pm \Delta f$, $\alpha \in \{\pm 1/2^p \pm 1/2^q\}$, $p, q = 0, 1, \ldots 8$.

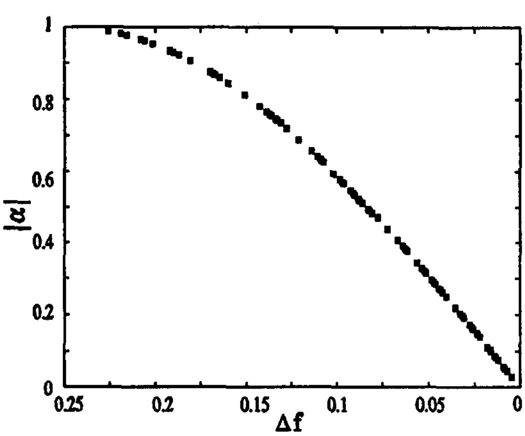

Figure 4. $f_{3dB} = 1/4 \pm \Delta f$, $\alpha \in \{\pm 1/2^p \pm 1/2^q \pm 1/2^r\}$, $p, q, r = 0, 1, \ldots 8$.

The procedure for adjusting the second group of coefficients is based on the sensitivity analysis of the transfer function realized by a parallel connection of two allpass networks, presented in the Appendix. From equation (A4), the magnitude response sensitivity can

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be computed as the product of the filter reflectance function, $|\sin((\phi_a - \phi_b)/2)|$, and the phase sensitivity of the corresponding first- or second-order section, $\partial\phi_i(\omega)/\partial x$. It is evident from (A3) and (A4) that the magnitude response sensitivity in the pass-band, where $(\phi_a(\omega) - \phi_b(\omega))/2 \approx 0$, is very low, whereas it is higher in the stop-band where $|\phi_a(\omega) - \phi_b(\omega)|/2 \approx \pi/2$. The transfer functions of these filters yield a very small a_p, what permits taking only the stop-band margin into account in adjusting β_i. For stop-band attenuation minima, $\max|\pi/2 - |\phi_a(\omega) - \phi_b(\omega)|/2|$, are calculated according to the specified margin, and the approximate values of coefficients β_i are then determined through a number of trials. The influence of a coefficient increases as the appropriate pole approaches the unit circle.

The sensitivity depends on the phase sensitivity to α and β as shown in the Appendix. It should be noticed that the influence of quantization of α is larger than the quantization of β, as shown in Figs. 5 and 6. Therefore, the coefficient α is determined to exact values without quantization and realized with small number of shift-and-add operations.

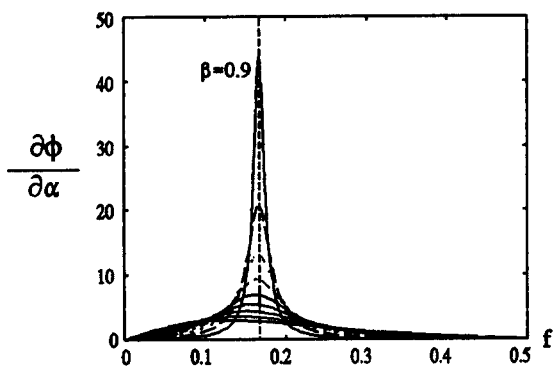

Figure 5. Phase sensitivity to α for $\alpha = -1/2$, $\beta \in \{0.1, 0.2, ..., 0.9\}$

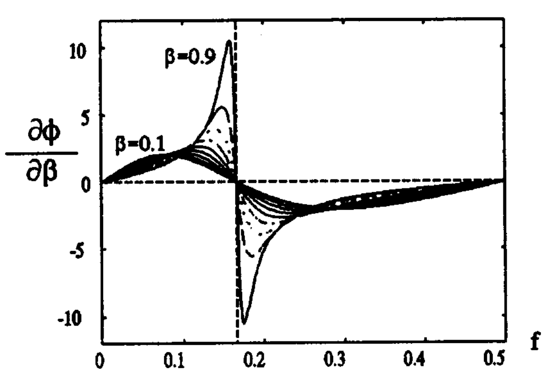

Figure 6. Phase sensitivity to β for $\alpha = -1/2$, $\beta \in \{0.1, 0.2, ..., 0.9\}$

4. APPLICATION

The adjustment of the common coefficient α is independent of the filter order and transition bandwidth. Practically, this means that $(n-1)/2$ coefficients can always be represented by a certain number of shifters and adders. The second group of coefficients is easy to adjust for the third-, fifth- and 7th-order filters. Explicit expressions for the third-order filters are derived.

Let start with known $\alpha_2 = \alpha$ and β_2. Then the auxiliary values are determined

$$\sigma_n = -\frac{1-\beta_2}{(1+\beta_2)\sqrt{(1-\alpha_2^2)}} \tag{9}$$

$$r = -\sigma_n(1+\sigma_n) + \sqrt{\sigma_n(\sigma_n+2)(\sigma_n^2-1)} \tag{10}$$

$$L = \sqrt{\frac{(1+2r)^3}{(1-r)^3\,(1+r)}}, \tag{11}$$

$$\Omega_a = \sqrt{\frac{1+2r}{(1+r)^3\,(1-r)}} \tag{12}$$

From the following relations, the minimal stopband attenuation and the edge frequencies can be determined:

$$a_a = 10\log(1+L) \tag{13}$$

$$\Omega_a = \frac{1-\alpha_2}{1+\alpha_2}\tan^2\pi f_a = \frac{1+\alpha_2}{1-\alpha_2}\frac{1}{\tan^2\pi f_p} \tag{14}$$

It is shown in Fig. 7 that a very good rather complex filters are obtained by a cascade connection of lower-order filters owing to the very small pass-band attenuation. They are suitable for the implementation of linear-phase IIR filters in accordance with [4].

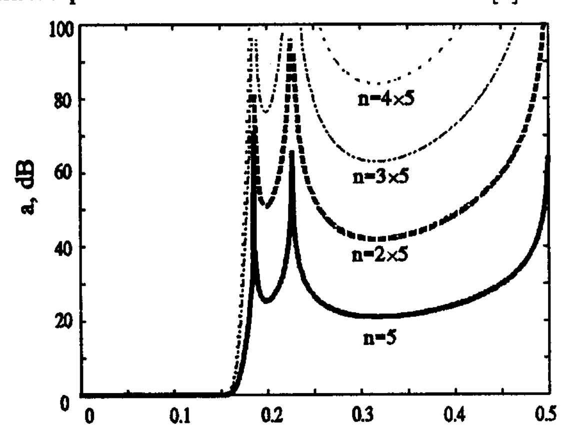

Figure 7. Attenuation of cascade connection of fifth-order filters α=-1/2, α_1=-1/4, β_2=1/2-1/32, β_3=1-1/8.

The procedure described is also applicable to half-band filters, since their transfer function is also included in this class of IIR filters [8]. In that case, the coefficients α equal zero, as the poles are on the imaginary axis. The procedure proposed in this paper can also be used successfully for the lattice wave digital filter structures from [9].

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Table 1. Multiplier coefficients (α and β) for f_{3dB}=0.5/3, $f_p \approx 0.153$, $f_a \approx 0.18$.

n	α	α_1	β_2	β_3	β_4	a_p(dB)	a_a(dB)
3	-1/2	-1/4	-1/4			0.238	11
5	-1/2	-1/4	1/2-1/32	1-1/8		0.0356	21
7	-1/2	-1/4-1/64	1/4+1/32+1/64	1-1/4-1/8+1/32	1-1/16-1/32	0.0026	30

Examples: Table 1 gives the data for 3 respective examples of the third-, fifth- and seventh-order filters for which f_{3dB}=0.5/3, $f_p \approx 0.153$, and $f_a \approx 0.18$.

As can be seen, a cascade connection of several simple filters from Table 1 can yield a high stop-band attenuation and a narrow transition band, with the pass-band attenuation remaining low. For example, for the linear phase filters from [4], the attenuation values listed in Table 1 are multiplied by two.

5. APPENDIX

The first-order sensitivity of the magnitude response $|S_x^{|H|}|$ is defined as a partial derivative to an arbitrary multiplication constant x

$$S_x^{|H|}(\omega) = \frac{\partial |H(e^{j\omega})|}{\partial x} \qquad (A1)$$

It is given in [10] that the filter magnitude response can be expressed by the phase difference $(\phi_a(\omega) - \phi_b(\omega))$:

$$|H(e^{j\omega})| = \frac{1}{2}\left|1 + e^{j(\phi_a(\omega)-\phi_b(\omega))}\right| \qquad (A2)$$

where $\phi_a(\omega)$ and $\phi_b(\omega)$ are the phases of allpass networks. Equation (A2) can be written in the form:

$$|H(e^{j\omega})| = \left|\cos\frac{\phi_a(\omega) - \phi_b(\omega)}{2}\right| \qquad (A3)$$

The application of (A1) to equation (A3) leads to the following expression for $S_x^{|H|}$:

$$S_x^{|H|} = \frac{k}{2}\left|\sin\frac{\phi_a - \phi_b}{2}\right|\left(\sum_i \frac{\partial\phi_{ai}(\omega)}{\partial x} - \sum_i \frac{\partial\phi_{bi}(\omega)}{\partial x}\right) \qquad (A4)$$

where $k = \text{sign}(\tan((\phi_a - \phi_b)/2)$, $\partial\phi_{ai}/\partial x$ and $\partial\phi_{bi}/\partial x$ are the phases of the ith first- or second-order sections whitch contains the multiplication constant x.

6. CONCLUSION

This paper presents a straightforward approach for multiplierless IIR elliptic filter design.

It is shown that a special class of elliptic filters, derived from elliptic minimal Q-factor analog prototype, has poles on a circle in the z plane. The implementation of those filters by a class of low-noise computationally efficient recursive digital filters, as parallel connection of two allpass sections, provides a unique property of multiplier coefficients. A half of multiplier coefficients are equal to a common constant that is only function of a single frequency f_{3dB}. By a selection of f_{3dB}, the frequency at which the attenuation is 3dB, between pass-band and stop-band edge frequencies in the transition band, a common constant (and a half of coefficient multipliers) can be designed with minimal number of shift and add operations. This way, the eliptic property with a very small pass-band ripple is obtained without quantization of a half of multipliers.

Using the sensitivity analysis presented in the paper, the reminding half of multipliers may be designed for a minimum shift and add implementation. This is easily achieved for 3rd, 5th and 7th order filters. Due to a very low passband ripple, the higher order filters of a very good quality can be formed by a cascade connection of lower degree filters. This way, a sharp multiplierless filter of higher degree is obtained.

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