

# NEW EXCHANGE RULES FOR IIR FILTER DESIGN\*

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## ABSTRACT

This paper introduces a new set of exchange rules for a Remez-like algorithm for the Chebyshev design of IIR digital filters. It is explained that the essential difficulty, in applying the Remez algorithm to rational functions, is that on some iteration, there may be no solution to the interpolation problem for which the denominator is strictly non-zero in the interval of approximation. Then the usual procedure for updating the interpolation points can not be applied. The new rules for updating the interpolation points address precisely this problem for the two-pole case. It is shown with examples that, when the Remez-like algorithm of Hofstetter *et al.* is applied to rational functions, there is a way to update the interpolation points so that the algorithm converges rapidly, even when poles arise in the interval of approximation.

## 1. INTRODUCTION

The Remez exchange algorithm for Chebyshev approximation by polynomials is successfully used for FIR filter design (in the Parks-McClellan algorithm [6]). Most notably, it converges rapidly from any initial reference set. On the other hand, the Remez algorithm for approximation by rational functions [7] is not guaranteed to converge from any initial reference set. This limits the usefulness of the rational Remez algorithm, and has prevented it from being more widely adopted. However, when it does converge, it exhibits a quadratic convergence rate as in the polynomial case. The failure of the rational Remez algorithm takes on a specific form, described below. This paper describes a new set of exchange rules (for the two pole case) that greatly enlarges the region of convergence of a rational Remez-like exchange algorithm. With the new exchange rules, convergence can be achieved from initial reference sets that are dissimilar from the extremal set of the optimal equi-ripple solution.

## 2. INITIAL REFERENCE SET SENSITIVITY

Recall that the polynomial Remez algorithm proceeds by iteratively performing the two steps:

1. Solve an interpolation problem over a specified set of points (the *reference set*).
2. Update the reference set according to a set of *exchange rules*.

The interpolation problem is equivalent to a linear system of equations (but efficient interpolation formulas can be employed [8]). The reference set is updated by locating the extremal points of the error function and selecting an appropriate subset thereof. The rational Remez algorithm proceeds in exactly the same manner. In the rational case, however, the interpolation problem is equivalent to a generalized eigenvalue problem [7], for which standard numerical techniques exist.

The essential difficulty in the rational case, is that on some iteration, there may be no solution to the interpolation problem for which the denominator is strictly non-zero in the interval of approximation. In that case, the solution to the interpolation problem contains singularities. Consequently the standard exchange rules can not be applied, and the algorithm can not be continued (see the figures). This is precisely the way the rational Remez algorithm fails.

When singularities appear during the course of the algorithm (i.e. not on the first iteration) it is sometimes possible to recover, by carrying out a systematic perturbation of the reference set used on the previous iteration [12, 13]. However, this recovery technique is not applicable when singularities appear on the first iteration, for there is no previous reference set to modify. The open problem, of how to continue in this case, is addressed below by a new set of exchange rules.

## 3. PREVIOUS WORK

The problem of best Chebyshev approximation for IIR digital filter design has recently been addressed by several authors, see [1, 4, 15] and the references therein. Methods that are not based on exchange algorithms tend to be computationally intensive, slow, and do not make use of the alternation theorem that characterizes optimal rational Chebyshev solutions (see [7] for a characterization theorem). Methods that employ exchange algorithms [1, 4, 5, 10, 11, 14, 15] avoid this initialization problem by alternating between passband and stopband approximation phases, by allowing at most one zero to contribute to the passband shaping, or do not address this problem.

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#### 4. PROBLEM FORMULATION

To simplify the approximation problem, we consider the design of *extra-ripple* lowpass recursive digital filters. In the symmetric FIR case, this subset of equi-ripple filters was the first for which a Remez-like exchange algorithm was developed; that algorithm is the algorithm of Hofstetter *et al.* [3]. Restricting our attention to this class of equi-ripple IIR filters simplifies the interpolation problem (the generalized eigenvalue problem is replaced by a linear system) and potential degeneracy problems are avoided [12, 13]. The design of this subclass of IIR filters is still affected by the problem of singularities in the solution to the interpolation step. Therefore, by considering this class of filters, we isolate the problem of singularities from other aspects of the approximation problem. It also produces extra-ripple IIR filters having more than one zero inside the unit circle, contributing to the shape of the passband, as in [10].

Consider the design of an IIR filter having seven zeros and two poles (away from the origin). We design the square magnitude response, and use the mapping  $x = \cos \omega$ , so that the problem formulation is as follows. Suppose  $\delta_p$  and  $\delta_s$  are given (specified by the user). Find

$$F(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{i=0}^7 p_i x^i}{1 + \sum_{i=1}^2 q_i x^i} \quad (1)$$

satisfying

$$F(x_i) = \begin{cases} 0 & i = 1, 3, 5 \\ \delta_s & i = 2, 4 \\ 1 & i = 6, 8, 10 \\ 1 - \delta_p & i = 7, 9 \end{cases} \quad (2)$$

where  $x_1, \dots, x_{10}$  are the extremal points of  $F(x)$  over  $[-1, 1]$  with  $x_1 = -1$  and  $x_{10} = 1$ . Let  $\omega_a = \omega_5$  and  $\omega_b = \omega_6$ ; these frequencies are the interpolation points that straddle the transition region. In the examples considered in this paper, we will use  $\delta_p = \delta_s = 0.05$ . It should be understood that this rather high value is used for illustrative purposes.

Given  $x_1, \dots, x_{10}$ , the interpolating function  $F(x)$  can be found by solving a linear system, or by using more efficient interpolation formulas. The algorithm of Hofstetter *et al.* computes the extra-ripple solution by iteratively updating the points  $x_1, \dots, x_{10}$ . On each iteration, those points are set equal to the new extremal points. The solution, shown in Fig. 1, is an extra-ripple filter having two zeros lying inside the unit circle, contributing to the shape of the passband. The remaining five zeros lie on the unit circle. The coefficients are  $B(z)/A(z) = (0.3368, 0.4674, 0.6155, 0.3967, 0.0950, 0.0047, -0.1037, 0.0747)/(1.0000, 0, 0.8871)$ . Because  $\delta_p = \delta_s$  and an equal number of points are assigned to the passband and stopband, for this example, the solution is a spectral factor of a (noncausal) halfband filter. Therefore, it could be used for the construction of an orthogonal IIR 2-channel filter bank.

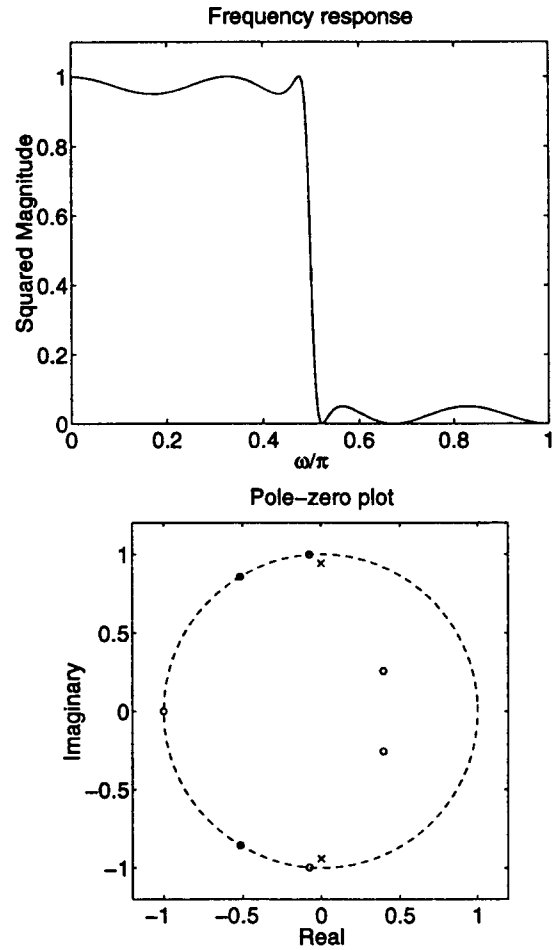


Figure 1. The extra-ripple IIR solution satisfying (1,2).

#### 5. NEW EXCHANGE RULES

Suppose the initial reference set is chosen to be a set of points uniformly spaced over  $[-1, 1]$ . The rational function  $F(x)$  satisfying the interpolation requirements has two singularities in  $[-1, 1]$ , see Fig. 2. In the figure, the circular marks indicate the interpolation points used to obtain  $F(x)$ . The vertical dashed lines indicate the location of the poles. Denote the two poles by  $p_1, p_2$ , with  $p_1 < p_2$ . The presence of these singularities precludes the continuation of the standard exchange algorithm. The exchange rules described below, however, provide a method by which the algorithm can be continued. It should be noted that singularities also occur in the rational Remez algorithm proper (where  $\delta$  is part of the interpolation problem at each iteration).

Note that in Fig. 2, the two poles lie between  $x_a$  and  $x_b$ . In this case it has been found that the appropriate update procedure updates  $x_a$  and  $x_b$  by:  $x_a \leftarrow p_1$  and  $x_b \leftarrow p_2$ . The remaining interpolation points are updated in the usual manner [3]. The new interpolation points are indicated by 'x' marks. On the next iteration,  $F(x)$ , computed using the new interpolation points, shown in the inset, has no singularities in  $[-1, 1]$ , and the usual exchange algorithm yields

the optimal solution in a few more iterations.

For a different initial reference set,  $F(x)$  is obtained as shown in Fig. 3. As in the preceding paragraph, the rule for updating the set of interpolation points is illustrated in the figure. On the following iterations, the usual exchange rules yield convergence in a few more iterations.

Figure 4 shows another initial reference set, the resulting function  $F(x)$ , and the proposed update rule. In this case, the next iteration also contains two poles in the transition region. The next iteration is shown in Fig. 5 where the update rule is illustrated. The usual update rules suffice for the following iterations, and convergence is quickly achieved. This example shows that more than one iteration is sometimes required before  $F(x)$  is free of poles in  $[-1, 1]$ .

Figure 6 illustrates a case where one of the poles lies away from the transition region. In this case, the rules for updating the reference set can be more subtle. It becomes necessary to consider two cases. In the first case,  $F(x) - \frac{1}{2}\delta_s$  does not possess a zero between the two adjacent interpolation points that straddle the pole, see Fig. 6. In the second case, it does, see Fig. 8. (Similarly for a pole in the passband, but use  $F(x) - (1 - \frac{1}{2}\delta_p)$ .) In the first case, it appears, by investigation of many examples, that the two interpolation points on either side of the pole should be kept fixed, and the remaining interpolation points be updated, as shown in Fig. 6. In the second case, the interpolation points should be shifted away from the transition region, as shown in Fig. 8.

The case illustrated in Fig. 7 indicates the appropriate exchange rule, when one pole lies in the transition band and the other pole lies outside  $[-1, 1]$ . Examples have shown us that, if the exchange rules for the cases where singularities are present, are not carefully formulated, the iterations can fail to converge to the sought solution.

Other cases, not covered here, are those cases where both poles occur away from the transition region. These cases occur when the initial reference set is even more dissimilar from the optimal extremal set, than is so in the examples given above. The case, where the number of poles exceeds two, also requires further investigation. Note that the number of poles must be even, in order to obtain an extra-ripple solution [4], otherwise, a pole must lie on the real line, which, when zeros also contribute to passband shaping, can not produce an extremal point.

## 6. CONCLUSION

This paper has discussed a rational version of the Remez-like exchange algorithm of Hofstetter *et al.*, for the design of IIR extra-ripple filters. Specifically, new exchange rules have been described, that are useful *even* when the interpolating rational function possesses poles in the interval of approximation on the first iteration.

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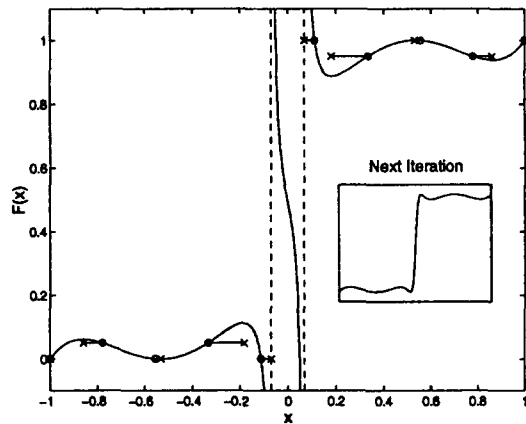


Figure 2. The rational function interpolating the circular marks possesses poles in  $[-1, 1]$ , indicated by the vertical dashed lines. When the interpolation points are updated as shown, the rational function shown in the inset is obtained. The usual exchange rules yield the extra-ripple solution in a few more iterations.

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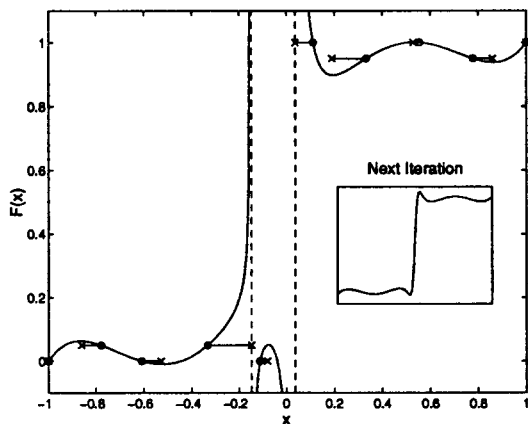


Figure 3. The usual exchange rules give the extra-ripple solution in a few more iterations.

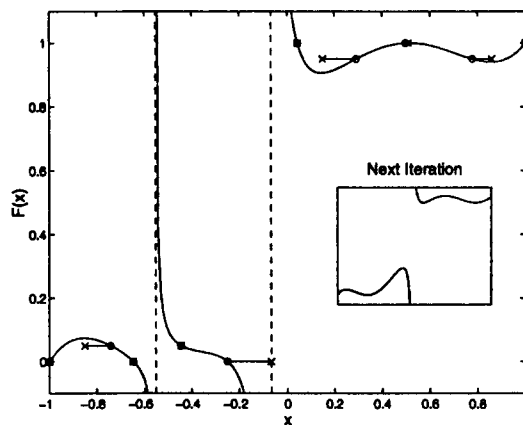


Figure 6. The next iteration has a single pole in  $[-1, 1]$ ; the other pole lies on the real line, outside  $[-1, 1]$ . The next iteration requires another update rule. It is shown in Fig. 7.

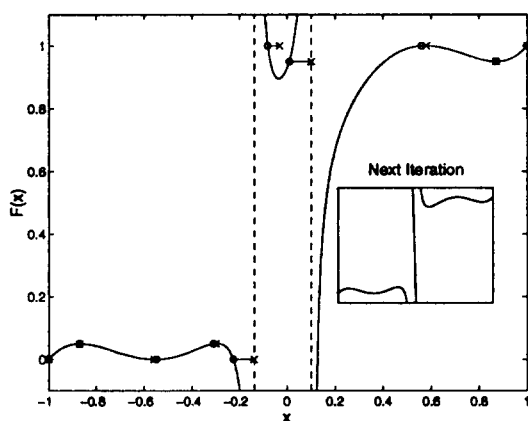


Figure 4. The next iteration requires another update rule. It is illustrated in Fig. 5.

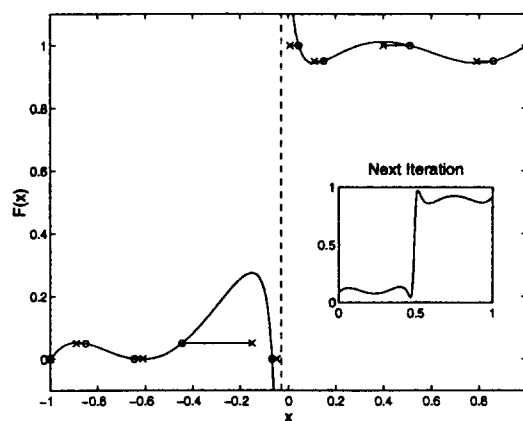


Figure 7. The usual exchange rules give the extra-ripple solution in a few more iterations.

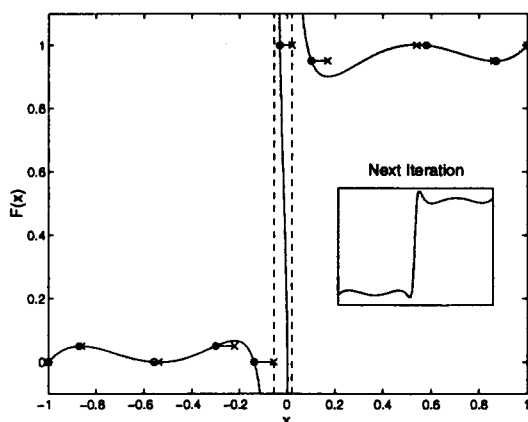


Figure 5. The usual exchange rules give the extra-ripple solution in a few more iterations.

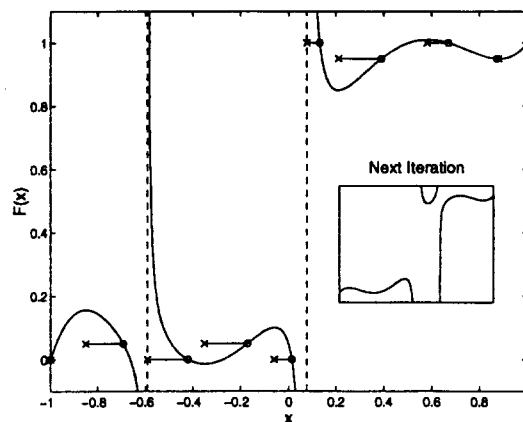


Figure 8. The update rules illustrated in the previous figures, when used on the next iterations, and the usual update rules on the following iterations, yield convergence in a few more iterations.