DESIGN OF RECURSIVE DIGITAL FILTERS WITH MAGNITUDE SPECIFICATIONS

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ABSTRACT

In this paper, we present an algorithm for the design of an optimal recursive digital filter with a specified magnitude frequency response. The method requires $O(N^2)$ computations to design a filter of size N, and exhibits numerical stability and quadratic convergence to the optimum within five iterations. The multiple exchange iterative algorithm uses the Chebyshev error criterion in the magnitude-squared frequency response domain, and has been developed using the interpolation theory and the alternation theorem for rational function approximation.

1. INTRODUCTION

Recursive digital filters are considered superior to nonrecursive filters in a variety of applications [5]. Generally, recursive filters can meet given frequency response specifications more efficiently than others.

An important subclass of problems in filter design is the case of magnitude response specification without phase consideration. In most cases, these specifications are described using the tolerance scheme. While practical methods for designing recursive filters for this problem exist for some specialized cases, a general solution has been lacking.

A number of approaches have been developed to design recursive digital filters with magnitude specifications using the tolerance scheme. Some of these methods introduce additional constraints to simplify the problem, thereby restricting their application [7, 9]. Other methods are either numerically unstable [2, 4] or computationally expensive [3, 6].

Recently, an $O(N^3)$ algorithm, for a filter of size N, was published [1]. The algorithm converges to the optimal solution, is numerically stable, and does not impose any artificial constraints; thus making it generally applicable and practical for moderate size problems. In this paper, we present an $O(N^2)$ algorithm that is a substantial modification to the published method, converges quadratically to the optimal solution and is numerically stable.

The rest of this paper is divided as follows: In Section 2, we describe the characterization of the solution and the major steps in our method. In Section 3, we present the core algorithm in our method that allows it to be $O(N^2)$. In Section 4, we present an example illustrating the method.

2. FILTER DESIGN

A filter design problem specified using the tolerance scheme is generally stated as follows: Given the ideal magnitude frequency response, $I(\omega)$, for $\omega \in \Omega \subseteq [0,\pi]$, and a specified tolerance; find the smallest recursive filter, with system function $H(z) = H_n(z)/H_d(z)$, that exceeds the specifications, where $H_n(z) = \sum_{m=0}^{N_n-1} h_n(m)z^{-m}$, and $H_d(z) = \sum_{m=0}^{N_d-1} h_d(m)z^{-m}$, $h_d(0) = 1$.

The filter has a magnitude-squared frequency response in the form of $\hat{H}(\omega) = B(\omega)/A(\omega)$, where $B(\omega) = \sum_{m=0}^{N_n-1} b(m)\cos(\omega m)$, $A(\omega) = 1 - \sum_{m=1}^{N_d-1} a(m)\cos(\omega m)$, and the filter size is $N = N_n + N_d$.

The desired filter is obtained as follows [1]: Transform the specifications to the magnitude-squared domain, resulting in the desired function $\hat{D}(\omega)$; compute the optimal coefficients for $\hat{H}(\omega)$; calculate a minimum-phase $H(z)$.

The optimal $\hat{H}(\omega)$ is one that minimizes the Chebyshev norm, $\| \hat{E} \| = \max_{\omega\in\Omega} |\hat{E}(\omega)|$, for given values of N_n and N_d. The weighted error, $\hat{E}(\omega)$, is defined as $\hat{E}(\omega) = \hat{W}(\omega)[\hat{D}(\omega) - \hat{H}(\omega)]$, $\omega \in \Omega$, where $\hat{W}(\omega)$ is the weighting function. The filter size is chosen to be the smallest that allows $\hat{H}(\omega)$ to exceed the specifications.

Although computations can be performed in the original domain, it is easier to transform the independent variable ω to $x = \cos(\omega)$, thereby yielding $\hat{H}(x) = D(x)/C(x)$, where $D(x) = \sum_{m=0}^{N_n-1} d_m x^m$, and $C(x) = 1 - \sum_{m=1}^{N_d-1} c_m x^m$. Relevant variables thus become functions of x, and parameters related to ω can be easily computed from those related to x.

Our top level algorithm to compute the optimal $\hat{H}(\omega)$ is based on the alternation theorem for rational approximation, which in our context states that the necessary and sufficient conditions for optimality is for $\hat{E}(x)$ to have N alternation points. In addition, the optimal solution is the best approximation over a discrete subset of size N [8]. The major steps of the top level algorithm are described below:

1. Choose N values of x, $x_1 < x_2 < \ldots < x_N$.
2. Compute the optimal filter over the N frequencies, and δ, as described later.
3. Determine $\hat{E}(x)$.
4. If $\max|\hat{E}(x)| \not\approx |\delta|$, find $x_1 < x_2 < \ldots < x_N$, the largest N local maxima of $|\hat{E}(x)|$ with alternation in sign and go to Step 2.
5. Otherwise, the filter designed in Step (2) is the desired filter.

The next section describes an $O(N^2)$ procedure to compute the optimal $\hat{H}(\omega)$ over the discrete set of N frequencies. It should be noted that the top level algorithm does not suffer from any degeneracies, even though it may superficially appear to have convergence problems. This is because theoretical issues related to degeneracy are not applicable in the filter design problem.

3. ALGORITHM

Here, we briefly state our procedure to compute the optimal $\hat{H}(x)$ over N fixed values of x, $x_1 < x_2 < \ldots < x_N$: Choose r to minimize the difference between $\hat{E}(x)$ and $(-1)^i r$ at $x = x_N$, where for a given r, $\hat{H}(x)$ is chosen so that $\hat{E}(x)$ satisfies $\hat{E}(x_i) = (-1)^i r$ for $i = 1, 2, \ldots, N-1$. A descent method for solving the one-dimensional minimization problem is utilized. The resulting r is denoted by δ.

An $O(N^2)$ procedure for computing $\hat{E}(x)$ at $x = x_N$ is summarized next. This procedure is based on Jacobi's method. In the following, $\lambda(x)$ denotes $(x - x_1)(x - x_2)\ldots(x - x_{N-1})$ and y_i denotes $\hat{D}(x_i) - (-1)^i r/\hat{W}(x_i)$ for $i = 1, 2, \ldots, N-1$.

1. Compute $c_1, c_2, \ldots, c_{N_d-1}$ by solving the set of equations: $\sum_{j=1}^{N_d-1} u_{mj} c_j = v_m$ for $m = 1, 2, \ldots, N_d - 1$, where $u_{mj} = \sum_{i=1}^{N-1} x_i^{N_d-1+j-m} y_i/\lambda'(x_i)$, and $v_m = \sum_{i=1}^{N-1} x_i^{N_d-1-m} y_i/\lambda'(x_i)$. Since this is a Toeplitz system, it can be solved in $O(N_d^2)$ computations.
2. Calculate $C(x_i), i = 1, 2, \ldots, N$. This is an $O(N^2)$ process.
3. Use the Lagrange interpolation formula to compute $D(x)$ at $x = x_N$ such that it satisfies the following: $D(x_i) = y_i C(x_i)$ for $i = 1, 2, \ldots, N-1$. This is an $O(N^2)$ procedure, and is conducted using the following equations:

$$w_i = \prod_{k=1, k\neq i}^{N-1} (x_i - x_k)$$

$$D(x) = \frac{\sum_{i=1}^{N-1} y_i C(x_i)/w_i(x - x_i)}{\sum_{i=1}^{N-1} 1/w_i(x - x_i)}$$

4. Calculate $\hat{E}(x)$ at $x = x_N$.

One should thus conclude that the above procedure is an $O(N^2)$ algorithm. Closer observation reveals that the overall method for recursive digital filter design is and $O(N^2)$ process, thus making our solution to the nonlinear problem computationally comparable to the Parks-McClellan method for the linear case. Also, all the elements in the method are numerically robust, thereby making the method stable.

4. EXAMPLE

In this section, we illustrate the algorithm using an example. A lowpass filter with passband cutoff frequency of 0.4π and stopband cutoff frequency of 0.6π is specified to have the magnitude frequency response within a tolerance of 0.005 in the passband and 0.01414 in the stopband.

In the magnitude-squared domain, the desired function, $\hat{D}(\omega)$, is unity in the passband and $1E-4$ in the stopband. The weighting function, $\hat{W}(\omega)$ is unity in the passband and 100 in the stopband. The required Chebyshev error, $\| \hat{E} \|$ is < 0.01.

A recursive digital filter, with $N_n = 5$ and $N_d = 5$ was designed with the indicated desired and weighting functions. The Chebyshev error of the resulting filter is 0.0084, Figure 1 shows the Chebyshev errors of filters designed at each iteration, and indicates that the optimal filter is computed by the fourth iteration. Figure 2 illustrates the weighted error function. $\hat{E}(\omega)$, of the designed filter. The nine alternations show that the filter is optimal. Figure 3 represents the magnitude frequency response in dB. The magnitude frequency response is larger than zero for all frequencies in the stopband because the filter specifications are exceeded. Figure 4 shows the behaviour of the magnitude frequency response in the stopband.

We have tested the method in other cases, and have noticed that the algorithm converges quadratically to the optimal solution in about five iterations.

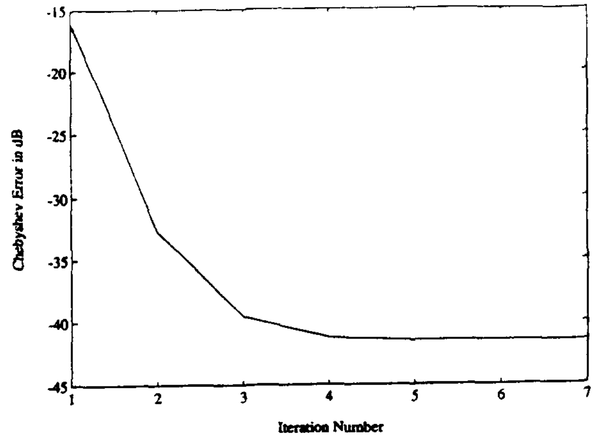

Figure 1. Convergence of the algorithm.

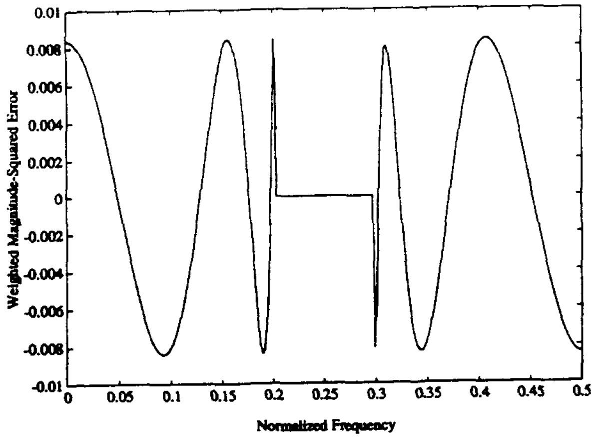

Figure 2. Weighted magnitude-squared error.

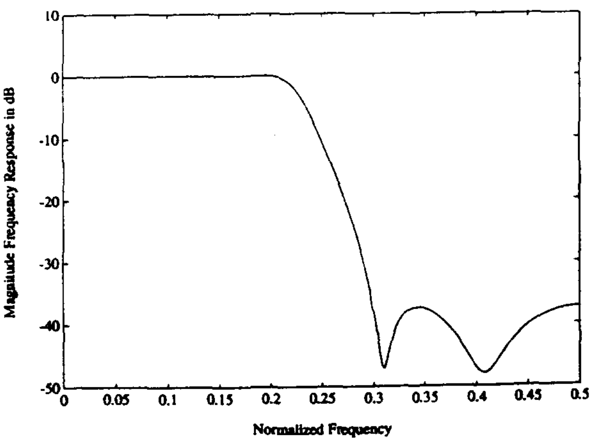

Figure 3. Magnitude frequency response in dB.

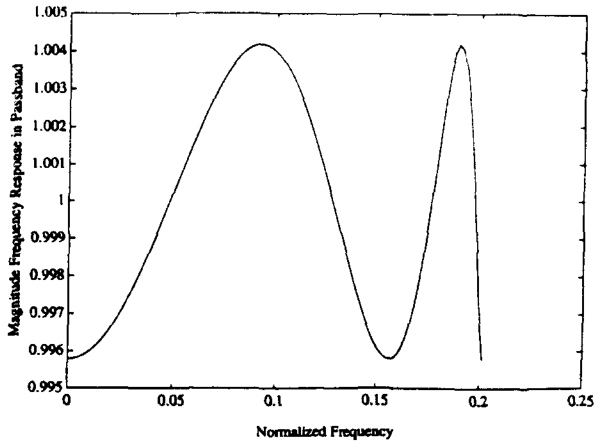

Figure 4. Magnitude frequency response in passband.

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