

# PERFORMANCE OF FRACTIONAL-DELAY FILTERS USING OPTIMAL OFFSET WINDOWS

*Anush Yardim<sup>\*</sup>, Gerald D. Cain<sup>\*</sup> and Arnaud Lavergne<sup>+</sup>*

<sup>\*</sup>University of Westminster, 115 New Cavendish St., London W1M 8JS, U.K.

<sup>+</sup>ESIEE Paris, Cite Descartes, BP 99, Noisy le Grand Cedex, France

E-mail: yardim@cmsa.westminster.ac.uk

## ABSTRACT

Windowing is an attractive avenue for creating asymmetric FIR impulse response sequences for use as fractional-delay elements. However, traditional (symmetric) windows are not usable, leading to a need for offsetting and purpose-optimization of the windows' defining parameters. Here it is shown that as few as three terms of such a modified raised cosine window deliver enormous improvements over simple sinc fractional-sample delays.

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Precision control of delay is important in many signal processing applications, particularly where signal rendezvous situations are encountered in two-channel systems or when direction of arrival to sensors must be measured accurately. Although it is a simple matter to inject delays amounting to integer multiples of the sampling interval being utilized, physical delays usually incur a residual extra portion which is an inconvenient fraction of a sampling interval. Fractional-sample delay filters (usually FIR in nature) utilize standard monorate signal handling in an attempt to perform the required signal interpolation. A superb overview of fractional-delay filters and the design issues involved can be found in [1].

Relatively little work has been done in harnessing windowing strategies for the fractional-delay task [2]-[6], despite their

simplicity and evident appeal when real-time updating of filter coefficients must be undertaken in order to track varying fractional delay. This is important because it is rare that the exact residual delay can be predicted for any physical signalling environment and, in any event, fluctuating physical transmission mechanisms will often lead to significant departure from nominal design assumptions (on the scale of a fractional-sample interval). Therefore, robustness and easy re-design/adaptability are at a premium in this application area. In this regard windowing clearly has a lot to offer since it is quick and flexible.

This paper re-visits the popular raised-cosine family of windows and elaborates the work in [6] in three directions:

- Extends to a three-term fully optimal window
- Confirms that variable fractional delay can be satisfactorily accommodated (for fixed commitment of filter dimensionality and measurement bandwidth) with an extremely simple window expression.
- Quantifies the shortfall experienced when simple approximations are employed to cut down optimization burdens when filter sizing and measurement conditions are allowed to sweep widely.

We start with a desired ideal transfer function

$$D(\nu) = e^{-j2\pi\beta\nu} \quad (1)$$

which our  $N$ -coefficient FIR filter must closely approximate. Here,  $\nu$  is normalized frequency and

$$\beta = \alpha + r \quad (2)$$

utilizes the customary mid-sequence delay measure  $\alpha=(N-1)/2$  and  $r$  is the fractional delay being sought. It is sufficient to focus on  $r \in [-0.5, 0.5]$ .

With  $k$  as the time index for our FIR coefficients and the popular starting point

$$h_s(k) = \text{sinc}(k - \beta), \quad k = 0, 1, \dots, N-1 \quad (3)$$

we have a crude fractional delayor which is serviceable, but far from worthy of a high quality designation. It can be massively improved by undergoing windowing:

$$h(k) = h_s(k)w(\beta; k) \quad (4)$$

Although standard windows (such as Hamming) can be brought into this role, their grounding in "linear-phase" conditions (and hence patterns symmetric about  $\alpha$ ) weakens their usefulness in this inherently "lop-sided sequence" context. Familiar symmetric windows need themselves to be subjected to desymmetrizing in order to yield best results [2]; such a class of "offset windows" is signified by the  $\beta$  dependence in the  $w$  symbol given in (4).

In Figure 1 we show the effects of two traditional windows (von Hann and Blackman-Harris) which have been offset, as well as two purpose-optimized windows which form the main thrust of this paper. We extract for measurement only the peak of the complex approximation of  $D(\nu)$  (in dB) encountered over a given frequency band. It is a fact of life that fractional-delay filters are very good in the vicinity of d.c. and inevitably bad at Nyquist frequency. Figure 1 shows this very well, with even the humble untreated sinc filter (top trace) plunging toward perfection near d.c. yet climbing toward that magnet of Nyquist-

frequency misbehaviour known as the "Tarczynski bound" [7], which equates to -1.84dB for the  $N=21$ ,  $r=0.2$  case depicted in Figure 1. Measurement bandwidth (MBW) represents the endpoint of the interval (commencing at the easy d.c. end) and is vitally important when quantifying the goodness of a fractional-delay filter. Clearly when MBW is made small (say 0.1) almost any windowing strategy can reap massive benefits. Offset von Hann provides about 47dB improvement over the bare sinc at MBW=0.1, and Offset Blackman-Harris does some 9dB still better than that!

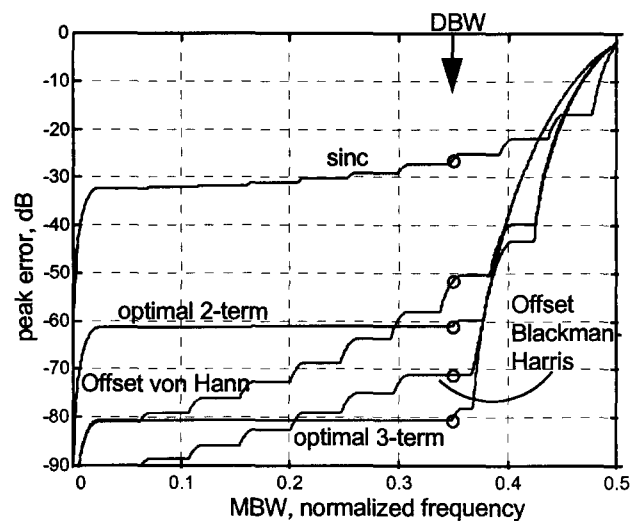


Figure 1 Error Profile for Several Delays ( $N=21$ ,  $r=0.3$ )

Demanding good behaviour from d.c. up to ever-higher MBW bandedges is where the difficulty comes in. The MBW value chosen as the edge of the constrained behaviour interval for optimization action is called the Design Bandwidth (DBW). Notice the long flat plateaus of peak error that our two custom-optimized windowed filters furnish, managing only slow, reluctant growth upward until their constraint DBW of 0.35 is met, (the MBW coinciding with DBW is indicated by circles) after which point their errors rocket out of control.

Our concern is with such controllable plateaus and raised-cosine windows which deliver them. The chosen class of windows is defined by

$$w(\beta;k) = \sum_{m=1}^I C_m \cos\left(\frac{2(m-1)\pi(k-\beta)}{N}\right) \tag{5}$$

with the only requirement being that the  $C_m$ 's sum to unity. Thus an optimal two-term window will require optimization only of  $C_1$ , while a three-term window ( $I=3$ ) requires  $C_1$  and  $C_2$  to serve as the optimization variables. The gap sizes in Figure 1 reflect the general truth: the best two-term window gives a gigantic improvement over the sinc delayor alone, and more (but relatively less) performance is bought by going to the best three-term window. We have found that numerical difficulties (using MATLAB's Optimization Toolbox and NAG Foundation Toolbox facilities) set in for higher  $I$ , making even reliable four-term  $C_m$ 's problematical. In this paper we will restrict consideration just to  $I=2$  and 3, favouring - of course - the three-term case for the excellence of results it can deliver.

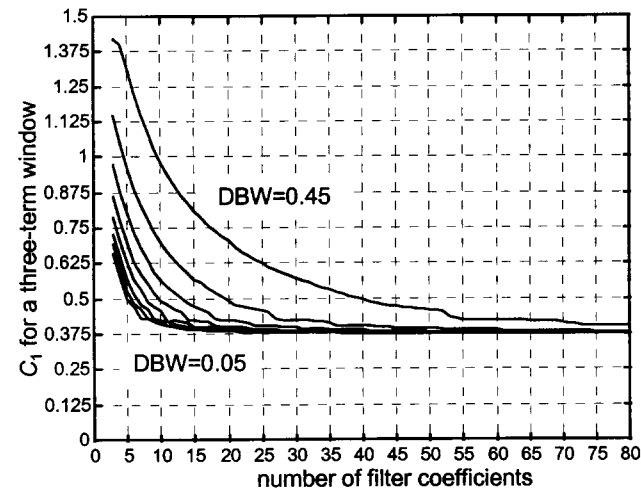


Figure 2  $C_1$  Variation with  $N$  ( $r=0.25$ )

Figure 2 illustrates the profound effect of DBW on just the  $C_1$  weight factor when  $I=3$ . In this view fractional delay is fixed, and we see clearly how demands for wide bandwidth fidelity (elsewhere [2] we have dubbed

DBW=0.45 "wideband" design and DBW=0.25 "halfband" design) inhibits convergence of  $C_1$  to its "natural" large- $N$  asymptotic value of 0.375.

We have found in the experiments underpinning this paper that (virtually independent of  $r$ ) the asymptotic window weight values are

$$\begin{aligned} I = 3 : C_1 &= 0.375, C_2 = 0.5, C_3 = 0.125 \\ I = 2 : C_1 &= 0.5 = C_2 \text{ (Offset von Hann)} \end{aligned} \tag{6}$$

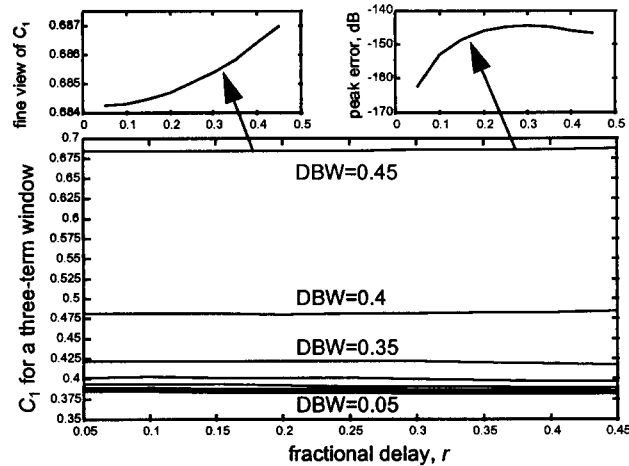


Figure 3  $C_1$  Variation with  $r$  ( $N=21$ )

Figure 3 examines variability, with fractional delay, for a situation (with  $N=21$ ) sized well below the large-filter asymptotes observed in many experimental runs. Again the DBW stratifications are familiar echoes of Figure 2; the leftmost inset is a blowup which shows very slight sensitivity to  $r$ . Such variation seems to be limited to the third decimal point of the  $C_m$ 's, comfortably below about 1% of their optimal values. This is a godsend in practical application scenarios, permitting us to optimize to a mid-range situation of  $r=0.25$  and then commence operation with no special regard to the  $\beta$  value being demanded by tracking/adaption procedures. Notwithstanding the slight  $C_m$  sensitivity, the peak error inset in Figure 3 discloses that there is still a peak error price (on the order of 18dB) to be paid. Disregarding this loss, we opt for operational

convenience where all we need do is change  $\beta$  in (3) and (4), but leave the  $C_m$ 's unperturbed!

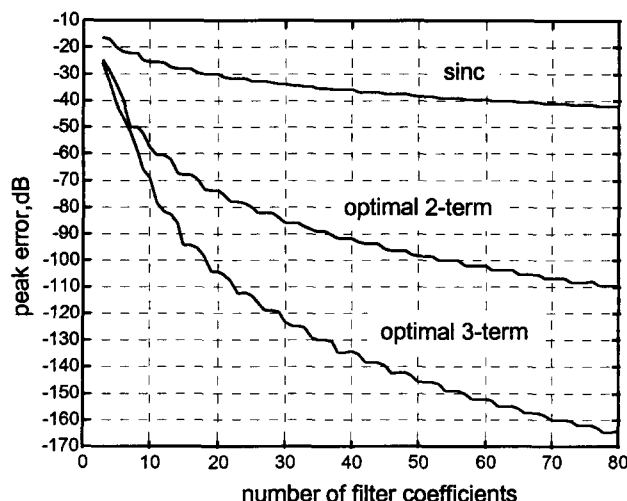


Figure 4 Error Improvements

Figure 4 exhibits the optimum peak error obtainable for a range of investments in filter dimensionality. Here we have adopted the "middle-of-the-road" parameters  $\text{DBW}=r=0.25$ , for a good representative snapshot of what can be achieved. Note, for instance, that a 50-coefficient delayor experiences some 60dB reduction of its peak error by virtue of optimal two-coefficient windowing, and deployment of a single additional optimal windowing weight secures a further 48dB of improvement! The net result is an easily-implemented scheme that performs in a manner comparable to delayors where considerably more computational effort (on all  $N$  filter coefficients) is needed in various alternative optimization strategies [1], [7].

In Figure 5 we take one further (but dramatic) step toward operational simplification, overlaying two additional curves onto Figure 4's plots: those arising from application of the highly streamlined situation (asymptotic) of (6). Although we incur inevitable sacrifices in performance (about 9dB for  $N=50$  in the two-term window case and about 21dB when three terms are used), the quality of filters obtained in this family is still extremely good. Thus, even

in this most "optimization-blind" simplification, offset windowing convincingly proves its worth.

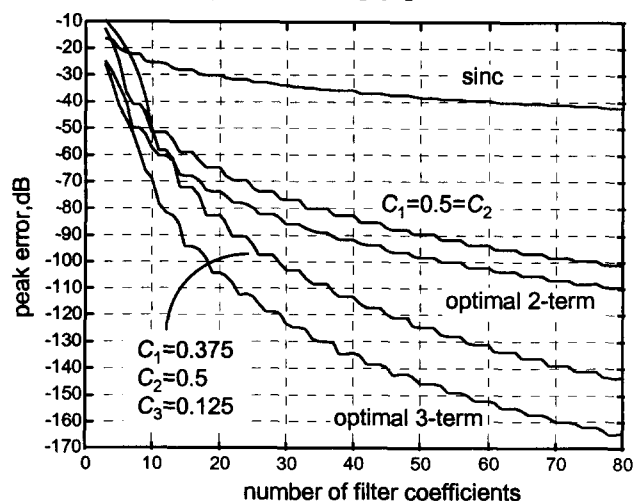


Figure 5 Easy Error Compromises

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