FIR FILTERING OF NONUNIFORMLY SAMPLED SIGNALS

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ABSTRACT

Filtering signals sampled on a grid which is nonuniformly distributed in the time domain is not a simple task since the filter's coefficients have to be time varying. They must be updated at each sampling instant. The filtering becomes even more complicated when it has to be optimal (or at least suboptimal) in the sense of a certain design criterion. In this paper we present an effective algorithm for FIR filtering aiming at minimisation of the energy of the filtering error signal. The approach provides a solution which resembles Weighted Least Squares design method for FIR filters of uniformly sampled signals.

1. INTRODUCTION

Discrete time signal processing concentrates, by and large, on dealing with signals which are registered on uniformly spaced time grids. Sometimes, however, there is a need for operating on irregularly sampled signals. There are two main reasons that such situations may occur:

- Firstly, due to technical problems, it is sometimes difficult or even impossible to perform regular sampling. If a signal is monitored on an "opportunity to measure" basis, like occasionally happens in astronomy or medicine, then the sampling grid is irregular and special processing techniques may be needed. Other examples include situations when a significant proportion of samples, of an otherwise uniformly sampled signal, is lost. This may happen with very noisy radar observations if, during pre-processing, some samples have to be discarded or with a digital signal stored on a faulty media. If not all samples can be recovered then one has to deal with a signal sampled on an irregular grid.
- Secondly, nonuniform sampling is sometimes intentionally used by system designers. The principal reason for doing so is to alleviate the frequency limits which normally must be satisfied by the signal spectrum in order to avoid aliasing. For an excellent review of benefits which may be brought by nonuniform sampling see [1]. Other recent publications on this topic include [2],[3].

The main goal of this paper is to generalise the Weighted Least Squares (WLS) approach, widely used in classical "uniform" DSP for designing FIR filters, so that it can be deployed for filtering signals sampled on a nonuniformly distributed grid. Similar problem but restricted entirely to uniformly sampled signals with lost samples has been solved in [4].

2. BASIC NOTIONS

We assume that all signals considered are baseband and bandlimited. Therefore there exists a bounded set Ω symmetric about zero which supports the spectrum of each signal. The bandwidth B of signals supported by Ω is

$$B = \int_{\Omega} d\omega \, . \tag{1}$$

The input to the designed filter is denoted by $x(t)$ and the spectrum of $x(t)$ is $X(\omega)$. $G(\omega)$ is the ideal frequency response of the filter. Hence the target spectrum of $y(t)$ - the filter's output - is

$$Y(\omega) = G(\omega)X(\omega) \, . \tag{2}$$

For practical reasons, it is assumed that at each stage of processing only a limited number of irregularly distributed samples of $x(t)$ is available. The sampling instants appropriate for these samples are denoted by $t_k, \cdots, t_{k-N}$. The nonuniform FIR filter operates on them according to the following equation

$$z(\tau) = a_{k\tau}x(t_k) + a_{k-1\tau}x(t_{k-1}) + \cdots + a_{k-N\tau}x(t_{k-N}) \tag{3}$$

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Notice that the coefficients $a_{k-i\tau}$ have to be updated each time τ changes and a new value of $z(\tau)$ is to be obtained. The nonuniform FIR filter is thus *time-varying*.

3. DERIVATION OF THE FIR FILTER

The filter's coefficients are selected in such a way that the magnitude of the absolute value of the difference $e(\tau)$ between $y(\tau)$ and $z(\tau)$

$$e(\tau) = y(\tau) - a_{k\tau}x(t_k) - a_{k-1\tau}x(t_{k-1}) - \cdots - a_{k-N\tau}x(t_{k-N}) \quad (4)$$

is kept small. The right hand side of (4) can be expressed with use of the inverse Fourier transform

$$e(\tau) = \frac{1}{2\pi}\int_\Omega X(\omega)\left(G(\omega) - \sum_{i=0}^{N} a_{k-i\tau}\exp(j\omega(t_{k-i} - \tau))\right) \times$$

$$\times \exp(j\omega\tau)d\omega = \frac{1}{2\pi}\int_\Omega X(\omega)F(\omega)\exp(j\omega\tau)d\omega \quad (5)$$

where

$$F(\omega) = G(\omega) - \mathbf{A}_{kN\tau}^T \mathbf{Z}_{kN}(\omega),$$

$$\mathbf{A}_{kN\tau} = \left[a_{k\tau}, \cdots, a_{k-N\tau}\right]^T$$

and

$$\mathbf{Z}_{kN}(\omega) = \left[\exp(j\omega(t_k - \tau)), \cdots, \exp(j\omega(t_{k-N} - \tau))\right]^T.$$

Now, we can evaluate $|e(\tau)|$ in the following way

$$|e(\tau)| \le \frac{1}{2\pi}\int_\Omega |X(\omega)F(\omega)|d\omega \le$$

$$\le \frac{\sqrt{B}}{2\pi}\sqrt{\int_\Omega |X(\omega)F(\omega)|^2 d\omega}. \quad (6)$$

The last expression in (6) has been derived using the Schwartz inequality. It follows from (6) that it is a reasonable approach to select the coefficients $a_{k-i\tau}$ in such a way that the weighted integral of the squared magnitude of $F(\omega)$: $\frac{1}{2\pi}\int_\Omega |X(\omega)|^2|F(\omega)|^2 d\omega$ is minimised. This formulation of the filter design problem closely resembles the WLS method for classical FIR filter design [5]. A major problem with proper use of the WLS approach is to determine the ideal weight $|X(\omega)|^2$. In most cases this will be replaced with an estimate since the spectrum $X(\omega)$ is rarely known accurately. Denote this estimate by $V(\omega)$, which must be a real, non-negative and even function. Note that the weight $V(\omega)$ should reflect the shape of $|X(\omega)|^2$ rather than its size. This is due to the fact that solutions to WLS design problems are identical regardless of whether $V(\omega)$ or $|g|V(\omega)$ $(g \ne 0)$ is used. Hence, our task is to determine the values of $a_{k-i\tau}$ $i = 0, 1, \cdots, N$ such that

$$J = \frac{1}{2\pi}\int_\Omega V(\omega)|F(\omega)|^2 d\omega \quad (7)$$

is minimised. The integral J is a quadratic function of vector $\mathbf{A}_{kN\tau}$: $J = \mathbf{A}_{kN\tau}^T\mathbf{M}_{kN}\mathbf{A}_{kN\tau} - 2\mathbf{A}_{kN\tau}^T\mathbf{K}_{kN\tau} + L$ where

$$\mathbf{M}_{kN} = \frac{1}{2\pi}\int_\Omega V(\omega)\mathbf{Z}_{kN}(\omega)\mathbf{Z}_{kN}^H(\omega)d\omega \quad (8)$$

and

$$\mathbf{K}_{kN\tau} = \frac{1}{2\pi}\int_\Omega V(\omega)G(\omega)\overline{\mathbf{Z}_{kN}(\omega)}d\omega. \quad (9)$$

In (9) $\overline{\mathbf{Z}_{kN}(\omega)}$ denotes the complex conjugate of $\mathbf{Z}_{kN}(\omega)$. The optimal value for $\mathbf{A}_{kN\tau}$ is thus

$$\mathbf{A}_{kN\tau} = \mathbf{M}_{kN}^{-1}\mathbf{K}_{kN\tau}. \quad (10)$$

4. REAL TIME IMPLEMENTATION

Filtering of nonuniformly sampled signals can be performed by using formulas (3) and (8)-(10). However, the computational burden associated with calculations involved is enormous. To ease this problem further analysis of (8)-(10) aimed at determination of a cheaper way of calculating $\mathbf{A}_{kN\tau}$ is necessary. Notice that by expanding the right hand side of (8) we find out that the element of the symmetric matrix $\mathbf{M}_{kN}$ in the r-th row in the s-th column is

$$\mathbf{M}_{kN}(r,s) = v(t_{k-r+1} - t_{k-s+1}) = v(t_{k-s+1} - t_{k-r+1}), \quad (11)$$

similarly by expanding (9) we obtain the r-th element of $\mathbf{K}_{kN\tau}$ as

$$\mathbf{K}_{kN\tau}(r) = v_G(\tau - t_{k-r+1}), \quad (12)$$

where $v(t)$ and $v_G(t)$ are the originals of $V(\omega)$ and $V(\omega)G(\omega)$ respectively. Hence, when the filter is implemented equations (11) and (12) replace (8) and (9). It is also possible to simplify calculation of $\mathbf{M}_{kN}^{-1}$ in (10). The matrix must be inverted each time the sequence of samples of the input signal is updated. There are two elementary changes to this sequence - rejection of the last sample and bringing a new one to the front of the sequence. In any case the inverse of matrix $\mathbf{M}_{kN}$ can be obtained as a simple update of the previous inversion. Here we provide the appropriate equations for updating $\mathbf{M}_{kN}$.

When a new sample has to be added to the sequence we have to create $\mathbf{M}_{k+1,N+1}^{-1}$ from $\mathbf{M}_{kN}^{-1}$. Notice that

$$\mathbf{M}_{k+1,N+1} = \begin{bmatrix} v(0) & \phi^T \\ \phi & \mathbf{M}_{kN} \end{bmatrix} \tag{13}$$

where

$$\phi = \begin{bmatrix} v(t_k - t_L) & v(t_k - t_{L-1}) & \cdots & v(t_k - t_{L-N+1}) \end{bmatrix}. \tag{14}$$

Therefore by using the matrix inversion lemma [6] we get

$$\mathbf{M}_{k+1N+1}^{-1} = \begin{bmatrix} \dfrac{1}{v(0) - \phi^T \mathbf{M}^{-1} \phi} & \dfrac{-\phi^T \mathbf{M}^{-1}}{v(0) - \phi^T \mathbf{M}^{-1} \phi} \\ \dfrac{-\mathbf{M}^{-1}\phi}{v(0) - \phi^T \mathbf{M}^{-1} \phi} & \dfrac{\mathbf{M}^{-1}\phi\phi^T\mathbf{M}^{-1}}{v(0) - \phi^T \mathbf{M}^{-1} \phi} + \mathbf{M}^{-1} \end{bmatrix}. \tag{15}$$

For the sake of simplicity matrix $\mathbf{M}$ in (15) denotes $\mathbf{M}_{kN}$.

In the case of rejection of the last sample we have to calculate $\mathbf{M}_{kN-1}^{-1}$ from $\mathbf{M}_{kN}^{-1}$. To do this we partition $\mathbf{M}_{kN}^{-1}$ as below

$$\mathbf{M}_{kN}^{-1} = \begin{bmatrix} \Gamma & \beta \\ \beta^T & \alpha \end{bmatrix} \tag{16}$$

and, again by using the matrix inversion lemma, we express $\mathbf{M}_{kN-1}^{-1}$ as

$$\mathbf{M}_{kN-1}^{-1} = \Gamma - \frac{\beta\beta^T}{\alpha}. \tag{17}$$

5. EXAMPLE

The proposed method of signal filtering is tested by simulation. The example is prepared in such a way that it also demonstrates the effect of alias suppression when nonuniform sampling is used. The task is to filter a signal whose spectrum stretches from 3 to 10 rad sec^{-1}. The filter should reject frequencies higher than 7 rad. sec^{-1}. The signal is sampled at randomly selected time instants using an additive random sampling scheme [7] with the average sampling frequency 15 rad. sec^{-1}. Notice that if the sampling rate was constant and equal to this value then aliasing would occur and no reasonable processing would be possible. A 20 tap FIR filter is used to perform filtering.

The spectrum support set for this problem is

$$\Omega = [-10 \quad -3] \cup [3 \quad 10]. \tag{18}$$

The ideal frequency response of the filter is selected as the following bandpass function

$$G(\omega) = \begin{cases} \exp(-j3.98\omega) & \text{if } 3 < |\omega| < 7 \\ 0 & \text{otherwise} \end{cases}. \tag{19}$$

The weight $V(\omega)$ is chosen as unity for all frequencies belonging to Ω. Therefore $v(t) = \dfrac{\sin(10t) - \sin(3t)}{\pi t}$ and $v_G(t) = \dfrac{\sin(7(t-3.98)) - \sin(3(t-3.98))}{\pi(t-3.98)}$. The signal $x(t)$ used in the simulation is

$$x(t) = 3\cos(4t) + \sin(5t) - 2\cos(9t). \tag{20}$$

If a perfect filter was applied to this signal then its output would be $y(t) = 3\cos(4(t-3.98)) + \sin(5(t-3.98))$. This ideal output is compared against the samples of the actual output of the filter generated on a very dense, uniform time grid with sampling rate of 70 rad. sec^{-1}. This simulation is repeated with a constant input sampling rate (again 15 rad. sec^{-1}) The results of both experiments are presented in Figure 1. Notice that the simulation results in the second experiment are affected very much by aliasing.

6. CONCLUSIONS

A novel method of FIR filtering of nonuniformly sampled signals has been presented. The algorithm has a compact form and provides effective solution for short and medium tap-length filters. The feasibility of the approach has been confirmed by simulation.

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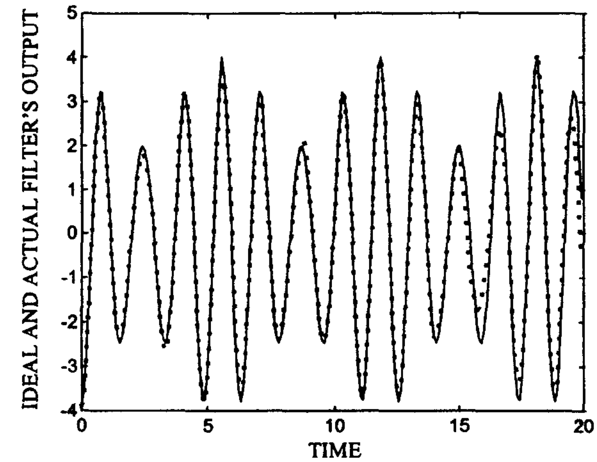
(a)



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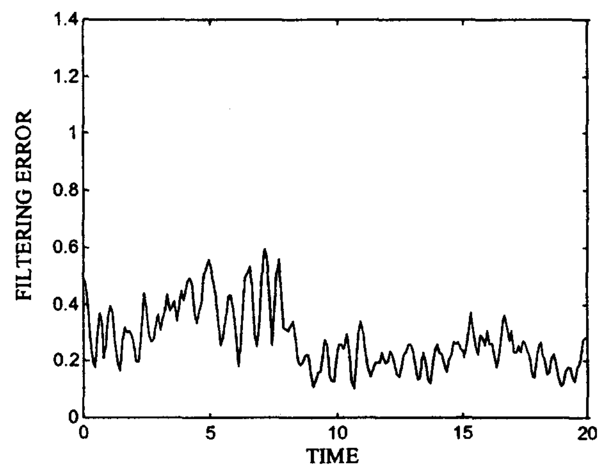

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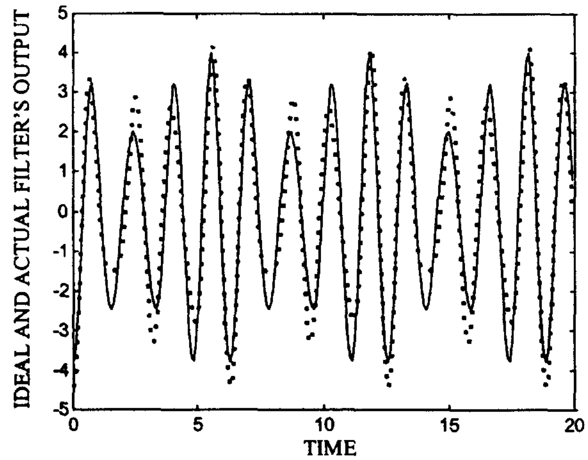

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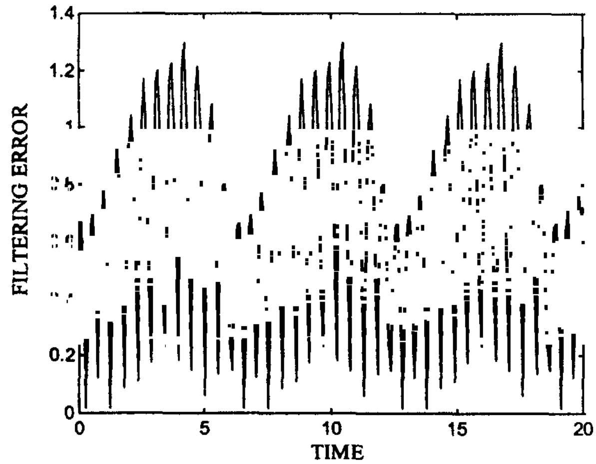

Figure 1.

(a) The “ideal” output of the filter (continuous line), and a typical output of the actual filter (dotted line)- nonuniform sampling case.

(b) Averaged (over 10 experiments) absolute value of the filtering error - nonuniform sampling case.

(c) The “ideal” output of the filter (continuous line), and the actual output of the filter (dotted line) - uniform sampling case.

(d) Absolute value of the filtering error - uniform sampling case.