# ROBUST STABILITY OF TIME-VARIANT DIFFERENCE EQUATIONS WITH RESTRICTED PARAMETER PERTURBATIONS: REGIONS IN COEFFICIENT-SPACE

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## ABSTRACT

Suppose rate of change of coefficients of a linear time-variant system modeled via a difference equation is restricted. The work presented herein is an attempt at developing an algorithm that determines regions in *coefficient-space* where such a system is guaranteed to be globally asymptotically stable. Such information can be extremely useful in many applications. Some previously published related results are consolidated as well.

## 1. INTRODUCTION

Robust stability of linear time-variant (LTV) systems with uncertain parameters have recently received considerable attention [1],[2],[3]. These results find important applications in adaptive signal processing and control, reconfigurable and hybrid systems [4],[5], etc., where the importance of the following problem [6] is clear: Given the rate of change of coefficients is bounded by, say $\Delta_{\max}$ (defined via an appropriate norm), find a region $\Omega_{\Delta_{\max}}$ in coefficient-space where global asymptotic stability (g.a.s.) of a time-variant (TV) system is guaranteed. Related work in [2] and [3] impose no limits on rate of coefficient variations which is not typically the case. For example in reconfigurable systems parameter changes are typically larger but their rates of change are restricted due to underlying system dynamics.

This work attempts at proposing a methodology for constructing such coefficient-space g.a.s. regions for LTV systems modeled via difference equations.

## 2. PRELIMINARIES

The following notation is used throughout: $\aleph_+$ denotes nonnegative integers; $\Re$ denotes reals; $\Re^m$ denotes the $m$-tuple vector space over $\Re$.

### 2.1. LTV Difference Equations

Consider the following LTV, finite dimensional, zero input, difference equation of order $m$:

$$y(n) = -\sum_{i=1}^{m} a_i(n)y(n-i) = -\mathbf{a}(n)^T\mathbf{y}(n-1), \quad (1)$$

where $\mathbf{a}(n) = [a_1(n), \ldots, a_m(n)]^T \in \Re^m$ and $\mathbf{y}(n-1) = [y(n-1), \ldots, y(n-m)]^T \in \Re^m$.

*Remarks:*

(a) In the linear time-invariant (LTI) case, the g.a.s. region $\Omega_{\mathrm{TI}}$ is determined via the conditions imposed by all roots of characteristic equation being inside unit circle in complex plane.

(b) Clearly $\Omega_{\Delta_{\max}} \subset \Omega_{\mathrm{TI}}$ and therefore $I - A(n)$ being singular for all $n$ is disallowed. For the linear system in (1), local and global asymptotic stability notions are equivalent.

**Theorem 1** **([2])** *Let* $\Omega_\diamond = \{\mathbf{a}(n) \in \Re^m : \|\mathbf{a}(n)\|_1 \leq \gamma < 1\}$. *Then, whenever* $\mathbf{a}(n) \in \Omega_\diamond$, $\forall n$, *(1) is g.a.s.*

For LTI case the g.a.s. region is $\Omega_{\mathrm{TI}}$. With no restrictions imposed on rate of rate of coefficient change, $\Omega_\diamond$ in fact denotes the largest hyperdiamond centered about origin within which coefficients may vary while ensuring g.a.s. [2].

Suppose rate of coefficient variations is restricted, viz., $\|\Delta\mathbf{a}(n)\| \leq \Delta_{\max}$ where $\Delta\mathbf{a}(n) = \mathbf{a}(n+1) - \mathbf{a}(n)$. Here $\|\cdot\|$ denotes any mutually consistent norm. Our objective is to determine a g.a.s. region $\Omega_{\Delta_{\max}}$ such that $\Omega_\diamond \doteq \Omega_\infty \subset \Omega_{\Delta_{\max}} \subset \Omega_{\mathrm{TI}} \doteq \Omega_0$. A previous attempt [7] possesses certain drawbacks: (a) Regions obtained do not properly contain $\Omega_\diamond$; (b) $\Delta_{\max}$ is not imposed *a priori;* (c) $\Delta_{\max}$ values implied are too conservative; (d) Computational burden is heavy.

### 2.2. Poles and Zeros of LTV Systems

For LTV systems Kamen [8] proposed certain TV poles and zeros. A necessary and sufficient condition that relates these to g.a.s. is also given. For simplicity, consider the second-order case of (1), that is,

$$y(n) + a_1(n)y(n-1) + a_2(n)y(n-2) = 0. \quad (2)$$

Define the two sequences $\{p_1(n)\}$ and $\{p_2(n)\}$ [8]

$$p_1(n) + p_2(n+1) = -a_1(n); \quad p_1(n)p_2(n) = a_2(n). \quad (3)$$

These are called the *left* and *right pole sets* of (2) respectively [8]. Define the Vandermonde matrix $V(n) = \begin{bmatrix} 1 & 1 \\ p_{2,1}(n) & p_{2,2}(n) \end{bmatrix}$, where $p_{2,1}(n)$ and $p_{2,2}(n)$ are a two right pole sequences computed from (3) for two different initial conditions. Then we have

**Theorem 2** **([8])** *Suppose that* $V(n_0)$ *is invertible. Then the LTV DT system in (2) is g.a.s. iff, for both* $i = 1$ *and* $i = 2$, $|p_{2,i}(n-1) \cdot p_{2,i}(n-2) \cdots p_{2,i}(n_0)| \to 0$ *as* $n \to \infty$.

Therefore a sufficient condition for g.a.s. is that, for both $i = 1$ and $i = 2$, $|p_{2,i}(n)| \leq \gamma < 1$, $\forall n > n_0$. These TV poles can be thought of as a more appropriate generalization of the usual TI ('frozen') poles.

## 3. CONSOLIDATION OF PREVIOUS RESULTS

For convenience results are described for the second-order case in (2); results for higher order case follows in a similar manner. From Theorem 2, a sufficient condition for g.a.s. of (2) is $\prod_{j=0}^{k-1} |p_{2,i}(n+j)| \leq \gamma < 1$, $\forall n > n_0$, for both $i = 1, 2$, that is, ensure that $k$ consecutive products are less than $\gamma$. However one only needs to ensure that consecutive products (*not necessarily of equal length*) are less than $\gamma$!

### 3.1. Two-Product Analysis

Use (3) recursively to get

$$\begin{aligned} p_2(n)p_2(n+1) &= -a_1(n)p_2(n) - a_2(n); \\ p_2(n+1) &= -a_1(n) - \frac{a_1(n)}{p_2(n)}. \end{aligned} \qquad (4)$$

In this work it is assumed that $p_2(n) \neq 0$. The case $p_2(n) = 0$ for some $n \in \aleph_+$ may be handled with only minor modifications [8]. Given $\gamma < 1$, construct the following subset of $\Omega_{\Delta_{max}}$:

$$\Omega^{(2)}_{\Delta_{max}} = \left\{ \mathrm{a} \in \Re^2 : |a_1(n)| + |a_2(n)| \leq \gamma < 1, \ \forall n \right\}. \qquad (5)$$

Of course $\Omega^{(2)}_{\Delta_{max}} = \Omega_\diamond$. Given that $\mathrm{a}(n) \in \Omega^{(2)}_{\Delta_{max}}$, $\forall n$, the following are obvious:

$$\begin{aligned} |p_2(n)| \leq 1 &\implies |p_2(n)p_2(n+1)| \leq \gamma < 1; \\ |p_2(n)| \geq 1 &\implies |p_2(n+1)| \leq \gamma < 1. \end{aligned} \qquad (6)$$

**Claim 1** *Suppose* $\mathbf{a}(n) \in \Omega^{(2)}_{\Delta_{max}}$, $\forall n$. *Then the following are true:*

*(a)* $|p_2(j)| \leq \gamma$, *for some* $j \in \aleph_+$
$\Rightarrow |p_2(j)p_2(j-1)| \leq \gamma$ *or* $|p_2(j-1)p_2(j-2)| \leq \gamma$.

*(b)* $|p_2(j)p_2(j-1)| \leq \gamma$, *for some* $j \in \aleph_+$
$\Rightarrow |p_2(j-1)| \leq \gamma$ *or* $|p_2(j-1)p_2(j-2)| \leq \gamma$.

*(c)* $|p_2(j)p_2(j-1)| \leq \gamma$, *for some* $j \in \aleph_+$
$\Rightarrow |p_2(j-1)p_2(j-2)| \leq \gamma$ *or* $|p_2(j-2)p_2(j-3)| \leq \gamma$.

Proof:

(a) Suppose $|p_2(j)| \leq \gamma$ and $\gamma < |p_2(j-1)p_2(j-2)|$. Because $|p_2(j-2)| \leq 1$ violates (6), we must have $1 < |p_2(j-2)|$. Now (6) implies $|p_2(j-1)| \leq \gamma \Rightarrow |p_2(j)p_2(j-1)| \leq \gamma$.

(b) Suppose $|p_2(j)p_2(j-1)| \leq \gamma$ and $\gamma < |p_2(j-1)|$. We must have $|p_2(j-2)| \leq 1$; otherwise (6) implies $|p_2(j-1)| \leq \gamma$ which is a contradiction. Now (6) implies $|p_2(j-1)p_2(j-2)| \leq \gamma$.

(c) Immediate from (b) and (c).□

**Lemma 3** *If* $\mathrm{a}(n) \in \Omega^{(2)}$, $\forall n$, *the LTV DT system in (2) is g.a.s.*

Proof: W.l.o.g take $|p_2(0)| \leq \gamma$. Pick an arbitrary integer $1 \leq N$. Apply (6) and Claim 1 to conclude that, w.l.o.g., we may take $|p_2(N)p_2(N-1)| \leq \gamma$.
Consider sequence $\{p_2(j)\}_{j=0}^{N}$. It is now possible to construct a sequence $\{q_2(i)\}$ where each $q_2(i)$ term has magnitude bounded by $\gamma$ and is a product of 1 or 2 consecutive terms of $\{p_2(j)\}$. Indeed an explicit strategy for this construction is as follows:

I. Put $i = 1$ and $j = N$.

II. Pick $q_2(i) = p_2(j)p_2(j-1)$.

III. If $|p_2(j-2)p_2(j-3)| \leq \gamma$, put $i = i+1$, $j = j-2$, and repeat Step II. Else pick $q_2(i+1) = p_2(j-2)$ and $q_2(i+2) = p_2(j-3)p_2(j-4)$; put $i = i+2$, $j = j-3$, and repeat Step III.

IV. Repeat this process until $p_2(2)p_2(1)$ or $p_2(1)$ are picked (in which case pick $p_2(0)$ as the last term of $\{q_2(i)\}$) or $p_2(1)p_2(0)$ is picked.

Therefore, given an arbitrarily small $\epsilon > 0$, we may make $\prod_{j=1}^{N} |p_2(j)| < \epsilon$ for sufficiently large $N$. Indeed choose $N > 2 \ln \epsilon / \ln \gamma$. Now Theorem 2 implies g.a.s. of (2).□

The fact that Theorem 1 is identical to restricting the $\infty$-norm of each consecutive two-product of the canonical state-space representation of (1) was shown in [7]. Notice that $\Omega^{(2)}_{\Delta_{max}} = \Omega_\diamond$. Hence Lemma 3 consolidates these norm based arguments and TV poles of [8].

*Remark:* Notice that no information regarding rate of coefficient change may be captured in two-product analysis.

### 3.2. Three-Product Analysis

Use (3) recursively to get

$$\begin{aligned} &p_2(n)p_2(n+1)p_2(n+2) \\ &\quad = [a_1(n)a_1(n+1) - a_2(n+1)]p_2(n) + a_1(n+1)a_2(n); \\ &p_2(n+1)p_2(n+2) \\ &\quad = [a_1(n)a_1(n+1) - a_2(n+1)] + \frac{a_1(n+1)a_2(n)}{p_2(n)}. \end{aligned} \qquad (7)$$

Let $\mathbf{a} = [a_1, a_2]^T$ and $\hat{\mathbf{a}} = [\hat{a}_1, \hat{a}_2]^T$. Define

$$X(\mathbf{a}, \hat{\mathbf{a}}) = |a_2\hat{a}_1| + |a_1\hat{a}_1 - \hat{a}_2|. \qquad (8)$$

Given $\gamma < 1$, consider the following subset of $\Omega_{\Delta_{max}}$:

$$\begin{aligned} \Omega^{(3)}_{\Delta_{max}} = &\left\{ \mathbf{a} \in \Re^2 : X(\mathbf{a}, \hat{\mathbf{a}}) \leq \gamma, X(\hat{\mathbf{a}}, \mathbf{a}) \leq \gamma, \right. \\ &\left. \forall \hat{\mathbf{a}} \in S(\mathbf{a}, \Delta_{max}) \cap \Omega^{(3)}_{\Delta_{max}} \right\}, \end{aligned} \qquad (9)$$

where $S(\mathbf{a}, \Delta_{max}) = \{\hat{\mathbf{a}} \in \Re^2 : \|\hat{\mathbf{a}} - \mathbf{a}\|_\infty \leq \Delta_{max}\}$. All point pairs $(\mathbf{a}, \hat{\mathbf{a}})$ must satisfy $X(\mathbf{a}, \hat{\mathbf{a}}) \leq \gamma$ and $X(\hat{\mathbf{a}}, \mathbf{a}) \leq \gamma$, $\forall n$; but they are not allowed to leave $\Omega^{(3)}_{\Delta_{max}}$. Its boundary $\partial[\Omega^{(3)}_{\Delta_{max}}]$ acts as a 'wall'; points located nearby may 'collide' on it but may not 'cross over' to its exterior! This is accounted for by including $\Omega^{(3)}_{\Delta_{max}}$ (which is exactly what needs to be found) in the right side of (9).

Consider the points in $\Omega^{(3)}_{\Delta_{max}}$. Then the following are obvious:

$$\begin{aligned} |p_2(n)| \leq 1 &\implies |p_2(n)p_2(n+1)p_2(n+2)| \leq \gamma < 1; \\ |p_2(n)| \geq 1 &\implies |p_2(n+1)p_2(n+2)| \leq \gamma < 1. \end{aligned} \qquad (10)$$

Using arguments similar to (but more cumbersome than) Claim 1 and Lemma 3, one may now show that clusters of 2 or 3 consecutive terms of $\{p_2(j)\}$ may be picked such that each cluster has a magnitude bounded by $\gamma$. Theorem 2 then implies g.a.s. of (2). It is easy to show that $X(\mathbf{a},\hat{\mathbf{a}}) \le \gamma$ and $X(\hat{\mathbf{a}},\mathbf{a}) \le \gamma$ is identical to restricting the $\infty$-norm of each consecutive three-product of the canonical state-space representation of (1). This consolidates the norm based arguments and TV poles of [8].

## 4. CONSTRUCTION OF G.A.S. REGIONS

How can we construct $\Omega^{(3)}_{\Delta_{\max}} \subseteq \Omega_{\Delta_{\max}}$ in coefficient-space $\mathbf{a} \in \Re^2$, given that $\|\Delta \mathbf{a}\| \le \Delta_{\max}$? Note that $\Omega_\diamond \Omega^{(2)}_{\Delta_{\max}} \subset \Omega^{(3)}_{\Delta_{\max}}$. Hence one objective of 'expanding' $\Omega_\diamond$ by incorporating rate of coefficient change is being met. The main cause of difficulty in computing $\Omega^{(3)}_{\Delta_{\max}}$ however is appearance of $\Omega^{(3)}_{\Delta_{\max}}$ itself in the right side of (9).

We propose to address this in two stages. The initial work was developed for the second-order case in (2); keeping computational complexity at a manageable level has nevertheless been a challenging task.

### 4.1. Far-From-Boundary (FB) Points

First consider *far-from-boundary (FB) points* defined thus:

$$\Omega^{(3)}_{\Delta_{\max},\mathrm{FB}} = \left\{\mathbf{a} \in \mathrm{int}[\Omega^{(3)}_{\Delta_{\max}}] : S(\mathbf{a},\Delta_{\max}) \subseteq \Omega^{(3)}_{\Delta_{\max}}\right\}, \tag{11}$$

where int[·] denotes interior. Each FB point needs at least two 'jumps' prior to 'colliding' with boundary. We first construct $\Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$.

**Observation 1** *Let* $\Delta_1 \le \Delta_2$. *Then* $\Omega^{(3)}_{\Delta_2} \subseteq \Omega^{(3)}_{\Delta_1}$. *In particular* $\Omega^{(3)}_{\Delta} \subseteq \Omega^{(3)}_{0}$, $\forall \Delta \ge 0$ . *Here* $\Omega^{(3)}_0 = \{\mathbf{a} \in \Re^2 : |a_1^2 - a_2| + |a_1 a_2| < 1\}$.

**Observation 2** *Suppose* $\mathbf{a}$ *is given. Then* $\mathbf{a} \in \Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$ *iff* $X(\mathbf{a},\hat{\mathbf{a}}) \le \gamma$ *and* $X(\hat{\mathbf{a}},\mathbf{a}) \le \gamma$, $\forall \hat{\mathbf{a}} \in S(\mathbf{a},\Delta_{\max})$.

Hence we may construct $\Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$ by first determining all $\mathbf{a}$ that satisfies the condition in Observation 2. To reduce this computationally prohibitive exhaustive search scheme, we have

**Lemma 4** *Suppose* $\mathbf{a}$ *is given. Then* $\mathbf{a} \in \Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$ *iff* $X(\mathbf{a},\hat{\mathbf{a}}) \le \gamma$ *and* $X(\hat{\mathbf{a}},\mathbf{a}) \le \gamma$, $\forall \hat{\mathbf{a}} \in \partial S(\mathbf{a},\Delta_{\max})$. *Here* $\partial[\cdot]$ *denotes boundary of region* $[\cdot]$.

Proof: Suppose $\mathbf{a} \in \Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$. Obviously $X(\mathbf{a},\hat{\mathbf{a}}) \le \gamma$ and $X(\hat{\mathbf{a}},\mathbf{a}) \le \gamma$, $\forall \hat{\mathbf{a}} \in \partial S(\mathbf{a},\Delta_{\max})$.
Conversely, suppose $\mathbf{a} \notin \Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$, that is, $\exists \hat{\mathbf{a}} \in S(\mathbf{a},\Delta_{\max})$ s.t. either $X(\mathbf{a},\hat{\mathbf{a}}) > \gamma$ or $X(\hat{\mathbf{a}},\mathbf{a}) > \gamma$. Reasoning is identical for both cases; hence take the former case, that is,

$$|a_2(a_1+\Delta a_1)| + |a_1(a_1+\Delta a_1) - (a_2+\Delta a_2)| > \gamma.$$

Clearly we must have either

$$\begin{aligned} |a_2(a_1+\Delta a_1)| + |a_1(a_1+\Delta a_1) - (a_2+\Delta_{\max})| &> \gamma, \text{ or} \\ |a_2(a_1+\Delta a_1)| + |a_1(a_1+\Delta a_1) - (a_2-\Delta_{\max})| &> \gamma. \end{aligned}$$

This implies $\exists \hat{\mathbf{a}} \in \partial S(\mathbf{a},\Delta_{\max})$ s.t. $X(\mathbf{a},\hat{\mathbf{a}}) > \gamma$ or $X(\hat{\mathbf{a}},\mathbf{a}) > \gamma$.□

Hence a scheme requiring less overhead to construct $\Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$ is as follows:

I. Pick $\mathbf{a} \in \Omega^{(3)}_0 \setminus \Omega^{(2)}_{\Delta_{\max}}$.

II. Check whether $X(\mathbf{a},\hat{\mathbf{a}}) \le \gamma$ and $X(\hat{\mathbf{a}},\mathbf{a}) \le \gamma$, $\forall \hat{\mathbf{a}} \in \partial S(\mathbf{a},\Delta_{\max})$. Actually proof of Lemma 4 implies that it is only necessary to check the two edges $\{\hat{\mathbf{a}} : \hat{a}_1 \in [a_1-\Delta_{\max}, a_1+\Delta_{\max}], \hat{a}_2 = \pm\Delta_{\max}\}$.

Results for $\|\Delta \mathbf{a}(n)\|_\infty \le \Delta_{\max} = 0.1$ (with $\gamma = 1 - 10^{-6}$) is in Figure 1. Given that $\|\Delta \mathbf{a}(n)\|_\infty$ does not vary at a rate

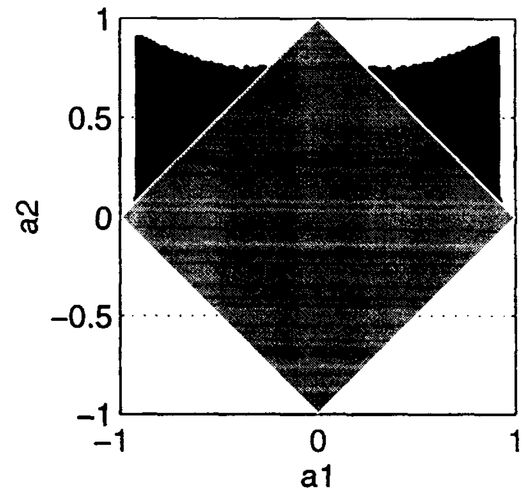


**Figure 1.** $\Omega^{(3)}_{0.1,\mathrm{FB}}$ **is denoted by light- and dark-shaded regions; light-shaded region denotes** $\Omega_\infty = \Omega_\diamond = \Omega^{(2)}_{\Delta_{\max}}$.

exceeding 0.1 (per time instant), the TV DT system in (2) is guaranteed to be g.a.s. as long as $\mathbf{a}(n)$ is restricted to be within $\Omega^{(3)}_{0.1,\mathrm{FB}}$. Significance and novelty of this information is worth mentioning:

(a) Maximum allowable rate of coefficient change is a priori imposed.

(b) Corresponding region is in coefficient-space.

(c) It is larger than any that is previously available.

### 4.2. Near-To-Boundary (NB) Points

Next consider *near-to-boundary (NB) points* defined thus:

$$\Omega^{(3)}_{\Delta_{\max},\mathrm{NB}} = \left\{\mathbf{a} \in \mathrm{int}[\Omega^{(3)}_{\Delta_{\max}}] : S(\mathbf{a},\Delta_{\max}) \cap \mathrm{ext}[\Omega^{(3)}_{\Delta_{\max}}] \neq \emptyset\right\} \tag{12}$$

where ext[·] denotes exterior. Note that $\Omega^{(3)}_{\Delta_{\max}} = \Omega^{(3)}_{\Delta_{\max},\mathrm{FB}} \cup \Omega^{(3)}_{\Delta_{\max},\mathrm{NB}}$. Given the appropriate 'direction,' each NB point needs only one 'jump' to 'collide' with boundary. To construct $\Omega^{(3)}_{\Delta_{\max},\mathrm{NB}}$ the following scheme may be used:

I. Put $\Omega^{(3)}_{\Delta_{\max},\mathrm{NB}} = \Omega^{(3)}_{\Delta_{\max},\mathrm{FB}}$.

II. Pick $\mathbf{a} \in \Omega_0^{(3)} \setminus \Omega_{\Delta_{max},NB}^{(3)}$.

III. If $X(\mathbf{a}, \hat{\mathbf{a}}) \leq \gamma$ and $X(\hat{\mathbf{a}}, \mathbf{a}) \leq \gamma$, $\forall \hat{\mathbf{a}} \in S(\mathbf{a}, \Delta_{max}) \cap \Omega_{\Delta_{max},NB}^{(3)}$, put $\Omega_{\Delta_{max},NB}^{(3)} = \Omega_{\Delta_{max},NB}^{(3)} \cup \{\mathbf{a}\}$ and repeat Step II with another $\mathbf{a}$; else repeat Step II with an alternate $\mathbf{a}$.

IV. Repeat this process until no further enlargement of $\Omega_{\Delta_{max},NB}^{(3)}$ is possible.

*Remarks:*

(a) The region obtained is a function of the particular order in which $\mathbf{a}$ in Step II were picked.

(b) Since the objective is to 'grow' the region $\Omega_{\Delta_{max},FB}^{(3)}$ previously obtained, points being picked in Step II should be 'just outside' (but still within $\Omega_0^{(3)}$). Therefore a 'boundary following' scheme is useful.

(c) Note that $\Omega_{\Delta_{max},FB}^{(3)}$ has no 'regular' shape (see Figure 1). Hence no simplification via any vertex/edge result akin to Lemma 4 is expected.

## 5. CONCLUSION

This work contains some preliminary work related to construction of regions in coefficient-space wherein a given LTV difference equation with restricted rate of coefficient change is guaranteed to be g.a.s.

For the $j = 2$ and $j = 3$ cases ($j = 1$ case is trivial), it has been shown that restricting $\infty$-norm of each consecutive $j$-product of the canonical state-space representation of (1) is identical to $j$-product analysis of TV poles in [8]. This in fact consolidates seemingly unrelated results that have appeared in the literature. Although no formal proof is yet available, it is conjectured that this relationship holds true for all $j \in \aleph_+$.

In constructing coefficient-space g.a.s. regions, only the second-order case in (2) has been addressed. An efficient algorithm (made possible by a certain edge result) to obtain $\Omega_{\Delta_{max},FB}^{(3)}$ has been developed. However, even for the second-order case, construction of $\Omega_{\Delta_{max},NB}^{(3)}$ can be time consuming. On the other hand, it is the authors' experience that, a significant extra region is not gained by computing the latter. Computation of $\Omega_{\Delta_{max},FB}^{(3)}$ can be efficiently done and it significantly improves the results in [2] (which does not incorporate rate of coefficient change) and [7] (the drawbacks of which were mentioned elsewhere).

### 5.1. Future Research Topics

Several interesting future research topics are worth mentioning:

(a) Further reduction of computational burden in computing $\Omega_{\Delta_{max}}^{(3)}$: Computational geometric reasoning seems a promising approach.

(b) Development of 'sufficiency' regions: Are there regions that are perhaps smaller than $\Omega_{\Delta_{max}}^{(3)}$ but are of more regular shape?

(c) Different norm restrictions on rate of coefficient change: Work presented above is related to $\infty$-norm. Different norms may provide easier algorithms.

(d) $n$-product ($n \geq 4$) analysis: With two-product analysis, maximum possible g.a.s. region is $\Omega_\circ$ itself; use of three-products allows one to extend this region to $\Omega_0^{(3)}$ (which is the limiting region obtained with $\Delta_{max} \to 0$). Use of $n$-products ($n \geq 4$) can provide even larger regions. However problem description and its solution is expected to be correspondingly more complicated.

(e) Extension to higher order systems: Again the solution is expected to be correspondingly more complicated.

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