OPTIMAL NOISE LEVELS FOR STOCHASTIC RESONANCE

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Abstract. **In a stochastic resonance system, additive noise and a nonlinear component system permit the amplification of a weak periodic signal, whenever the strength of the noise is within a certain interval. For one such a system with a nonlinearity consisting of a threshold function, we define a measure of goodness and, for the case of Gaussian noise, we derive required intervals of noise variance for stochastic resonance.**

I. INTRODUCTION

Under the paradigm of *Stochastic Resonance* [1]-[5], a class of models for physical systems that produce an approximately periodic output when driven by a weak periodic signal embedded in white additive noise, is studied. Common characteristics of the models are the presence of a weak periodic driving signal, a nonlinear memoriless component system and additive noise. In order for the output to have a periodic behavior, the strength of the noise should be within a certain *interval;* this last requirement is probably the reason for the name of the paradigm.

II. A STOCHASTIC RESONANCE SYSTEM

Consider the system in Fig. 1, with inputs the periodic driving signal s_n and noise r_n, and output t_n. r is white, zero mean and t is a thresholded version of s+r. Assume s to be a binary square (or approximately square) periodic discrete signal with peak amplitudes A and -A, period N and fundamental angular frequency $\omega_0 = 2\pi/N$. The signal s is weak in the sense that A > 0 is less than the threshold value U; only with the addition of the noise r, the output t may be non zero. Too much noise will obscure the periodicity of the output; too little noise will not make the output non zero. Thus, the variance of the noise should be within a certain interval.
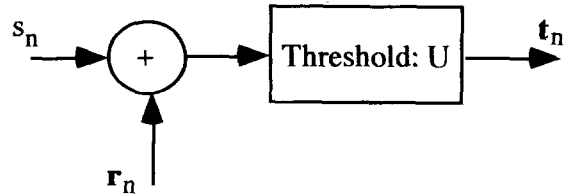

Fig.1. A Stochastic Resonator

Seen from another point of view, the signal t_n is a noisy version of the signal s_n where the noise (not r) is multiplicative and signal dependent, this makes difficult to obtain statistics of t_n analytically.

II. RESONANCE

In an ideal situation, we have $t_n = \text{sgn}(s_n)$. Since errors arise when A + r < U and when -A + r > U, we define the following measure of goodness for the system:

$$G = P(A + r > U) - P(-A + r > U).$$

Let F and f be the distribution and density probability functions of r, when r has unitary variance. In the case when the standard deviation is σ we may write,

$$G(\sigma) = F(\frac{U+A}{\sigma}) - F(\frac{U-A}{\sigma}) \qquad [1]$$

For zero-mean noise with even f, F(0) = 1/2 and using U − A > 0 we have that G is bounded above by 0.5. In Fig. 2, assuming A = 1 and for several values of U, we plot G versus σ, for Gaussian noise. We say that the system is in stochastic resonance whenever G is at least 25%; then, for A = 1, U must be less than 1.94. For a state of stochastic resonance an interval of values of σ is required: for U = 1.5, the interval is given by, [1.05, 3.8], for U = 1.7, [1.51, 3.53], for U = 1.9, [2.19, 2.99] and for for U = 1.935, [2.49, 2.72], etc.

The curves in Fig. 2 can be used for the cases of a signal strength A different from one: to a threshold level U in Fig. 2 there corresponds an actual threshold U' = UA,
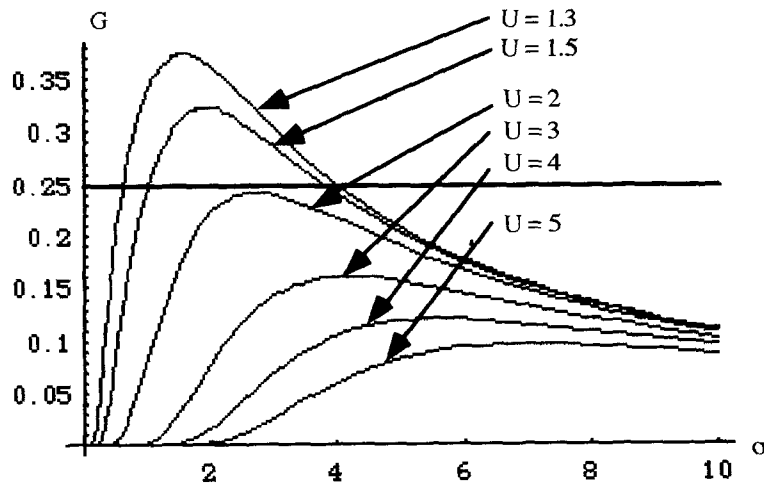
Fig. 2. The measure of goodness, as a function of σ. Note how the maximum of the curves shifts to the right as U increases.

and to a deviation σ in Fig. 2 there corresponds an actual deviation $\sigma' = \sigma A$.

III. OPTIMAL NOISE LEVELS

Given the signal amplitude A and the threshold level U, to find an optimal noise level σ that maximizes G, we differentiate Eqn. [1] and equate to zero obtaining,

$$\frac{1}{\sigma^2} [(U-A)f(\frac{U-A}{\sigma}) - (U+A) f(\frac{U+A}{\sigma})] = 0$$

Solving the equation above for the particular case of a Gaussian distribution, we obtain,

$$\sigma_{opt} = \sqrt{\frac{4AU}{\ln\frac{U+A}{U-A}}} \quad , U > A \quad\quad [2]$$

We have the following bounds for σ_{opt}:

$$\sqrt{2U(U-A)} < \sigma_{opt} < \sqrt{2} \; U$$

For U much larger than A, σ_{opt} behaves approximately linearly and independent of A:

$$\sigma_{opt} \approx \sqrt{2} \; U$$

which is a useful result: the optimal standard deviation of the noise for resonance grows linearly with the threshold level, independently of the amplitude of the signal.

The system is not supposed to operate with strong signals that is values of U close to A, nevertheless, we have for U close to A,

$$\sigma_{opt} \approx \frac{2A}{\sqrt{\ln\frac{1}{U-A}}}$$

IV. THE STATISTICS OF THE OUTPUT

The output process is not stationary but *cyclostationary* [6]. It consists of the repetition of white random vectors that is, $\{t_n\} = \; ... \; T_1, T_2, T_1, T_2 \; ...$ where $T_1 = [B, \; B \; ... \; B]$ and $T_2 = [C, \; C \; ... \; C]$; the random variables in T1 being the thresholded version of $A + r$ and of $-A + r$ in T2. Let M and $N - M$ be the lengths of T1 and T2, respectively. Denoting the thresholding function, with threshold level U, by T_U,

$$B = T_U(A + r) \quad \text{and} \quad C = T_U(-A + r)$$

assuming that the output of the threshold function takes values in $\{-1, 1\}$ and letting,

$$p_1 := P(B = -1) = 1 - P(B = 1) = F(U - A)$$
$$p_2 := P(C = -1) = 1 - P(C = 1) = F(U + A),$$

the correlation function of the output is given by,

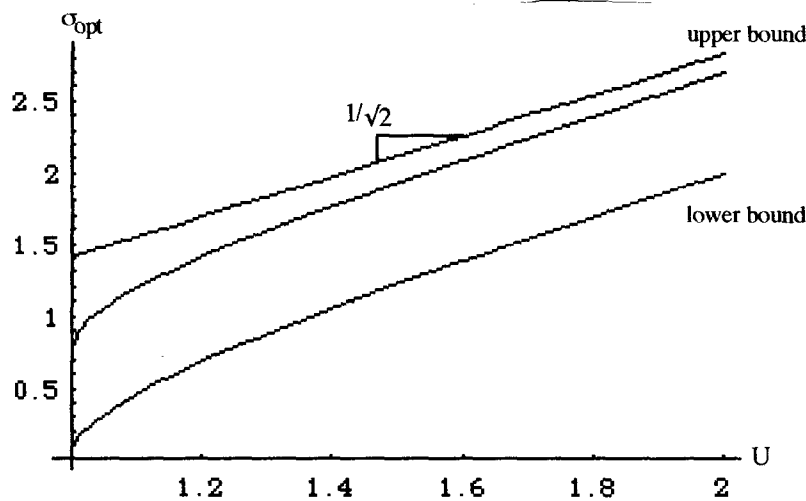Fig.3. Optimal noise level, as a function of the threshold level U.

$R(n, k) = E[t_n\ t_{n+k}]$

$= 1\ if\ k = 0,$

$= E^2[B] = (1 - 2p_1)^2,$

$if\ 0 \leq [n]_N \leq M - 1,\quad 0 \leq [n+k]_N \leq M - 1,\ k \neq 0,$

$= E^2[C] = (1 - 2p_2)^2,$

$if\ M \leq [n]_N \leq N - 1,\quad M \leq [n+k]_N \leq N - 1,\ k \neq 0,$

$= E[B]E[C] = (1 - 2p_1)(1 - 2p_2),\ otherwise.$

V. APPLICATIONS

We are currently exploring two adaptive stochastic resonance systems. First, for the simulation of a biological system that checks for the presence of a weak periodic signal within a certain frequency band, the threshold level is to be adjusted until there is a signal with large energy at the output of the band pass filter in Fig. 4. In the simulations we ran, for $\sigma \in [1, 1.4]$ and $A = 1$, the energy as a function of U peaks around $U = 1.5$; for frequencies $2\pi/N$ within the band pass the energy is about ten times the energy for frequencies off the band pass. Currently we are working on an analytical characterization of the output.

Also, we have considered systems where the strength of the noise is modified according to several estimators of the periodicity of t, see Fig. 5. In one case, we estimate the period as the distance between clusters of ones, stopping when the sample variance of such distances is small. In another, based on the histograms of the distances between consecutive 1's and between consecutive -1's. In the simulations, for the cases of N = 8, 12 and 16, we obtain estimates of N with an average error less than 4%. As above, we are working on the derivation of analytical expressions for the statistics of the output of the system

VI. CONCLUSIONS

For the stochastic resonance system of Fig. 1, we have defined a state of resonance and define a measure of the quality of such resonance. We have given bounds and approximations for the optimal strength of the noise, for stochastic resonance. We have obtained an expression for the correlation function of the output process, which is cyclostationary. In simulations of potentially useful stochastic resonance adaptive systems, we have found
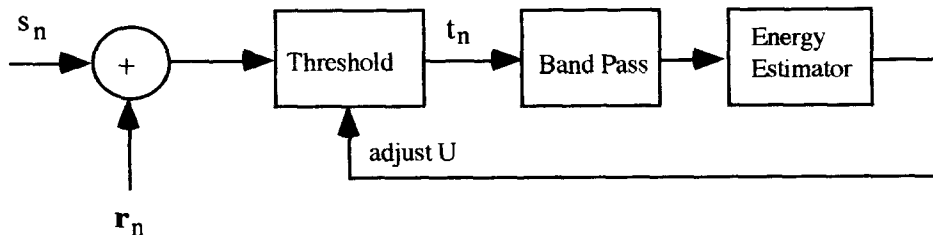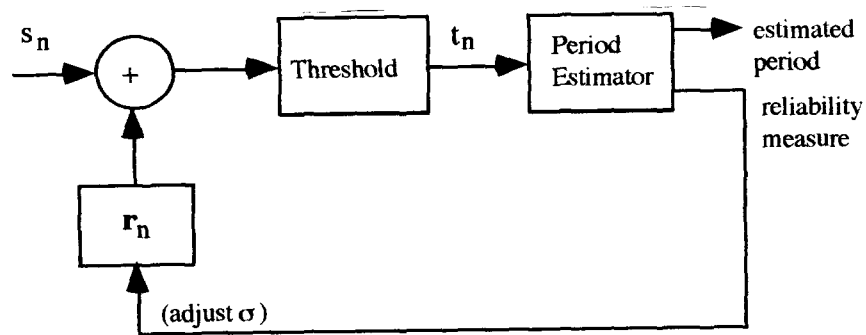

Fig. 4. Adaptive adjustment of threshold level U.

Fig. 5. Adaptive adjustment of noise strength σ.

agreement with the theoretical measures of goodness. We are currently working on the derivation of further analytical results, with the aim of making more precise and usable the paradigm of stochastic resonance, at least in some cases.

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