

# IMPROVED ACCURACY IN THE SINGULARITY SPECTRUM OF MULTIFRACTAL CHAOTIC TIME SERIES

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## ABSTRACT

Existing algorithms for accurately estimating the  $f(\alpha)$  singularity spectrum from the samples of generalized dimensions  $D_q$  of a multifractal chaotic time series use either linear interpolation of the known  $D_q$  values or finely sample the  $D_q$  curve. Also, the derivative in the expression for Legendre transform necessary to go from  $D_q$  to  $f(\alpha)$  is approximated using first order centered difference equation. Finely sampling the  $D_q$  is computationally intensive and the crude linear approximations to interpolation and differentiation give erroneous end points in the  $f(\alpha)$  curve. We propose using standard min-max filter design methods to more accurately interpolate between known samples of the  $D_q$  values and evaluate the Legendre transform. We use optimum (min-max) interpolators and differentiators designed with the Parks-McClellan algorithm. The new min-max approach exhibits computational reduction and improved accuracy. Examples are provided that show improved accuracy for attractors that contain multifractal behavior.

## 1. INTRODUCTION

Chaotic signals which can result from non-linear dynamical systems are very useful in many engineering applications, e.g. secure communications [1]. In signals analysis, these signals can be used to model natural phenomena [2].

There are several useful quantitative measures of chaos. One of these is the infinite hierarchy of generalized dimensions (GD)  $D_q$  [3] which provide useful information about the underlying dynamical system. They are defined as

$$D_q = \lim_{\epsilon \rightarrow 0} \frac{1}{1-q} \frac{\log(\sum_{i=1}^{M(\epsilon)} [p_i(\epsilon)]^q)}{\log(\epsilon)} \quad -\infty \leq q \leq \infty \quad (1)$$

where  $p_i(\epsilon)$  is the relative number of trajectory points in the  $i^{th}$  cell of size  $\epsilon$  and  $M(\epsilon)$  is the number of volume elements of size  $\epsilon$  needed to cover the attractor. If the attractor of a dissipative system has a non-integer correlation dimension  $D_2$ , then the system has a strange attractor. A chaotic attractor with non-constant GD is called a multifractal. Multifractal measures are used to determine various fractal regimes present in the chaotic attractor since two chaotic attractors with the same correlation dimension  $D_2$  can be quite different visually.

Although, theoretically, the  $D_q$  in (1) are known for all values of  $q$ , in practice, they are only computed for a finite

range  $q \in [q_{min}, q_{max}]$  and along a sampled grid with spacing  $\Delta q$ , i.e.,  $q = q_{min}, q_{min} + \Delta q, q_{min} + 2\Delta q, \dots, q_{max}$ . Large record lengths are needed to compute  $D_q$  when  $|q| \gg 0$ . Even when the recorded data lengths are long enough to give valid  $D_q$ , the most efficient box-counting algorithm [4] for computing each value of  $D_q$  has a computational complexity of  $\mathcal{O}(N \log N)$ , where  $N$  is the (large) length of the time series. Hence, generally, the number of computed samples of  $D_q$ ,  $\frac{|q_{max} - q_{min}|}{\Delta q}$ , is kept small by making the range  $|q_{max} - q_{min}|$  small or by coarsely sampling  $D_q$  using large  $\Delta q$ . In the next section, we consider the relationship between the generalized dimensions  $D_q$  and the  $f(\alpha)$  singularity spectrum.

### 1.1. $f(\alpha)$ singularity spectrum

When  $f(\alpha)$  and  $D_q$  are smooth functions of  $\alpha$  and  $q$ , the  $D_q$  in equation (1) are related to the singularity spectrum  $f(\alpha)$  by the Legendre transform [5, 6] in (4) given below:

$$\tau(q) = (q-1)D_q \quad (2)$$

$$\alpha(q) = D_q + (q-1)\frac{d}{dq}D_q \quad (3)$$

$$f(\alpha) = f(\alpha(q)) = q\alpha(q) - \tau(q) \quad (4)$$

Thus, if we know  $D_q$ , we can find  $\alpha(q)$  and  $f(\alpha)$  from equations (3)-(4) and vice versa.

The  $f(\alpha)$  singularity spectrum provides a mathematically precise and naturally intuitive description of a multifractal attractor in terms of interwoven sets with singularity strength  $\alpha$  whose Hausdorff dimension is  $f(\alpha)$  [5]. For any multifractal attractor, whose generalized dimensions are monotonic decreasing,

$$D_q > D_{q'} \quad \text{for } q < q', \quad (5)$$

the singularity spectrum  $f(\alpha)$  will be convex, with a single maximum at  $q = 0$  equal to  $D_0$ , i.e.  $f(\alpha(0)) = D_0$ , and with infinite slope at  $q = \pm\infty$  [5]. The extrema  $\alpha$  values where  $f(\alpha) = 0$  are termed  $\alpha_{min}$  and  $\alpha_{max}$ . They are the scaling indices for the densest and most rarefied areas in the attractor and they give the asymptotic values of  $D_q$  in (1), i.e.  $\alpha_{min} = D_\infty$  and  $\alpha_{max} = D_{-\infty}$ . Typically,  $D_{-\infty}$  and  $D_\infty$  are almost impossible to extract using equation (1) on real data because they require infinite length data records.

Ideally, we want the Legendre transform of the  $D_q$  curve in (4) to produce a very clean  $f(\alpha)$  curve with stable, converging end points. However, coarsely sampled  $D_q$  values

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produce errors in  $f(\alpha)$  [7]. Also, when the differentiation needed for  $\alpha(q)$  and  $f(\alpha)$  in (3)-(4) is not accurately implemented, the transform gives rise to an  $f(\alpha)$  envelope with spurious and erroneous end points. This may lead to ambiguous determination of  $D_\infty$  and  $D_{-\infty}$  [8].

## 2. PREVIOUS ALGORITHMS

A number of algorithms have been proposed to address the problem of accurate evaluation of  $f(\alpha)$ . These algorithms include the following: (1) The development of a minimal spanning tree algorithm [8]. This is an efficient algorithm for finding the nearest neighbors of any arbitrary point on the attractor. The method was proposed by Dominguez-Tenreiro et al. as an alternative to the generalized correlation integral (GCI) methods (equation (1)) of estimating the generalized dimensions such that the Legendre transform of the estimated  $D_q$  will give  $f(\alpha)$  with no spurious end-points. (2) Other methods [9, 10] have used variations of GCI in (1), to compute  $D_q$  except that in order to obtain an essentially continuous  $D_q$  curve needed for the Legendre transform, either (i) the known  $D_q$  values were linearly interpolated or (ii) the  $D_q$  values were computed at finely sampled points along the  $D_q$  curve using very small  $\Delta q$ . Also, derivatives in the expressions for the Legendre transform were approximated by first order centered difference equations.

These methods can be computationally intensive, give spurious end-points in  $f(\alpha)$ , and are sometimes only well suited to certain chaotic attractors [8]. Some of the major drawbacks of earlier methods are the introduction of spurious interior points in  $f(\alpha)$  caused by errors introduced in the linear interpolation of known  $D_q$  values and by approximating a derivative using a first order centered difference equation in the expressions for  $\alpha(q)$  and  $f(\alpha)$  in (3)-(4). Linear interpolation corresponds to a filter with a sinc-squared type frequency response [11] which can be a very poor approximation to the ideal "box" lowpass filter needed for interpolation. The centered first order difference operation can produce a poor sinusoidal approximation to the ideal  $j\omega$  frequency response of a differentiator.

In the next section, we describe the min-max filter design approach to the problem. The method involves the design of optimum lowpass filters and differentiators which are close to the ideal frequency responses needed in the accurate estimation of the  $f(\alpha)$  singularity spectrum.

## 3. MIN-MAX FILTER DESIGN

Given any multifractal time series and coarsely sampled values of  $D_q$  estimated from equation (1) in the finite range  $q \in [q_{min}, q_{max}]$ , we interpolate between these points by a factor  $L$ . This is accomplished by "stretching the signal", i.e. inserting  $L-1$  zeros between each pair of given  $D_q$  samples [12]. The effect of "stretching" the  $D_q$  is to create  $L-1$  mirror images of the original Fourier transform spectrum. To remove the mirror images, we pass the stretched signal spectrum through a lowpass filter whose passband edge is a function of  $L$  and the bandwidth of the original Fourier transform of the  $D_q$ . Thus, zero insertion between  $D_q$  samples followed by lowpass filtering reduces to computing the

convolution of the lowpass filter coefficients  $h[n]$  with the  $D_q$  curve with zeros inserted [12].

The coefficients of the lowpass or interpolation filter are readily obtained from the well known Parks/McClellan Remez Exchange filter design algorithm [13]. The program is available for designing linear phase finite impulse response (FIR) filters based on the Chebyshev approximation criterion. The Remez exchange algorithm finds the filter which minimizes the maximum weighted error in the passband and stopband, hence the term "min-max" optimization. The program may be used to design lowpass, highpass or bandpass filters, and differentiators [14]. The maximum errors in the passband and stopband are user-specified as  $\delta_1$  and  $\delta_2$ , respectively. For example,  $\delta_2 = 0.01$  corresponds to 40dB attenuation in the stopband. The Parks/McClellan algorithm gives the optimum (min-max) length  $n$  lowpass filter. If the initial frequency response of the filter does not satisfy the above specifications, then the filter order  $n$  is increased until the desired frequency response is attained.

## 4. RESULTS AND DISCUSSION

We applied the min-max filter method to synthetic and real generalized dimension data to check the versatility of our approach. In our examples, we have normalized the frequency axis by  $\frac{2\pi}{\Delta Q}$  so that  $F = 0.5$  corresponds to the Nyquist frequency  $\omega_N$ .

### Synthetic data:

The first example is that of the attractor of a Cantor set [2] that is asymptotically well modeled by a generator with two intervals of length  $r_1 = 0.408$  and  $r_2 = r_1^2$ , and with equal probability  $p_1 = p_2 = 0.5$ . The closed form expression for the GD,  $D_q$ , of the Cantor set is known [2]. This  $D_q$  expression was finely sampled ( $\Delta q = 1$ ) for  $q$  in a large range of -36 to 36, as shown in Fig. 1(a), to represent an ideal scenario. The corresponding singularity spectrum  $f(\alpha)$  in Fig. 1(b) was evaluated using a 6<sup>th</sup> order min-max differentiator designed assuming passband cut-off frequency  $F_p = 0.15$  and stopband  $F_s = 0.25$ , and error tolerances  $\delta_p = 0.1$  and  $\delta_s = 1 \times 10^{-6}$ . Notice that the maximum value for  $f(\alpha)$  in Fig. 1(b) corresponds to  $D_0$  in Fig. 1(a) and the two extrema  $\alpha_{min}$  and  $\alpha_{max}$  where  $f(\alpha) = 0$ , give the correct asymptotic values  $D_\infty = 0.387$  and  $D_{-\infty} = 0.773$  [2]. Next, in order to compare our new method with conventional algorithms under realistic conditions when few  $D_q$  values are available, the ideal  $D_q$  curve in Fig. 1(a) is purposely downsampled, i.e. evaluated at only 19 relatively coarsely sampled ( $\Delta q = 4$ ) values of  $q$ . Fig. 2(a) shows the linearly interpolated  $D_q$  values. Fig. 2(b) shows the  $f(\alpha)$  singularity spectrum using the linear interpolated values and centered first order difference operation. Notice the spurious end points as well as invalid interior points. In comparison, Fig. 2(c) shows the result of the min-max filter method on the sparsely sampled  $D_q$  using a 20<sup>th</sup> order min-max interpolator and 6<sup>th</sup> order min-max differentiator. The lowpass interpolation filter has the specifications  $F_p = 0.003$ ,  $F_s = 0.246$ ,  $\delta_1 = 0.1$  and  $\delta_2 = 1 \times 10^{-6}$ , the differentiator was designed to approximate an ideal  $j\omega$  frequency response between normalized frequencies 0 and 0.1. There are no invalid interior points in the  $f(\alpha)$  singularity spectrum and the spurious end points have been greatly re-

duced. The results of our method in Fig. 2(c) and the ideal case depicted in Fig. 1(b) are very close except at the ends of the curve, where  $f(\alpha) < 0$ . However, these errors are worse in the conventional approach.

#### Speech data:

The second example is the generalized dimensions,  $D_q$ , of real data, specifically, 4150 samples of an unvoiced fricative speech sound "S", spoken by an American female speaker in the ISOLETE database [15]. The generalized dimensions  $D_q$  were estimated in (1) using only 7 integer values ( $\Delta q = 1$ ) for  $q$  from -3 to 3 [7]. These 7 samples were optimally (min-max) interpolated by a factor of 4 and plotted in Fig. 3(a) along with the linear interpolation of the sampled  $D_q$  values in solid line. The figure clearly shows the poor approximation of linear interpolation when  $q$  is between 0 to 1 and 2 to 3. Fig. 3(b) shows the resulting  $f(\alpha)$  using linear interpolation and centered first order difference operation. Note the spurious and interior end points, which create an ambiguity in extracting information from the  $f(\alpha)$  curve. Our method, which used an efficient 20<sup>th</sup> order min-max interpolator and a 6<sup>th</sup> min-max differentiator, was able to estimate  $D_{-\infty}$  and  $D_{\infty}$  at  $f(\alpha) = 0$  from only 7 values of GD estimates. In this case, the lowpass filter has the specifications  $F_p = 0.004$ ,  $F_s = 0.2457$ ,  $\delta_1 = 0.1$  and  $\delta_2 = 1 \times 10^{-6}$ . The differentiator passband was 0 and 0.1.

In Fig. 3(c), the asymmetric spread of points in  $\alpha$  values around the maximum of  $f(\alpha)$  reveal the inhomogeneity in the attractor of unvoiced speech signals just like the variation in the  $D_q$  values [7].

Cubic spline interpolation was also implemented in comparison with the min-max approach, but cubic splines produced  $f(\alpha)$  singularity spectrum envelope with spurious and erroneous points when the attractor is marked by phase-transitions. Cubic spline interpolation can be a poor approximation of an ideal lowpass filter for signals that are not highly oversampled. Phase-transitions in a chaotic attractor occur when the otherwise smooth  $D_q$  curve has a discontinuous derivative [6].

## 5. CONCLUSION

The importance of this new approach to singularity spectrum calculation is its improved accuracy and its computational reduction. This is due to the fact that the  $D_q$  curve can be coarsely sampled and later optimally interpolated to obtain the smooth curve that is needed to compute the  $f(\alpha)$  singularity spectrum. The proposed method also provides estimates of the generalized dimensions at  $D_{\infty}$  and  $D_{-\infty}$  which are almost impossible to obtain using equation (1) on real data with limited number of data points. The method works well for attractors that exhibit multifractal behavior and satisfy equation (5) in the range between  $q_{min}$  and  $q_{max}$ . The proposed method exhibits improved accuracy over conventional linear or cubic spline interpolation.

## 6. REFERENCES

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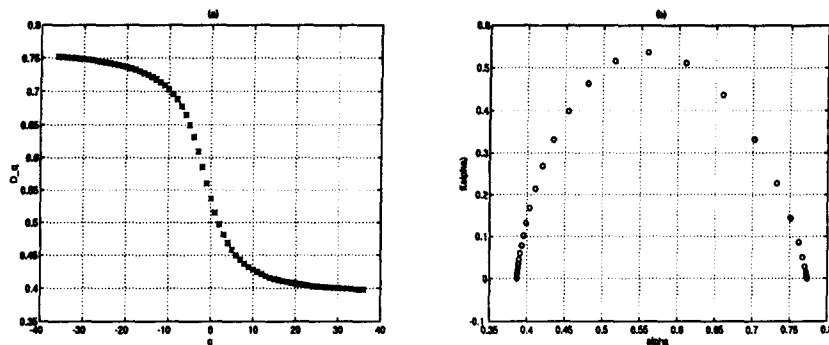


Figure 1: (a) Original GD,  $D_q$ , of the Cantor set, finely sampled ( $\Delta q = 1$ ) from a closed form expression in [2]. (b) Corresponding  $f(\alpha)$  Spectrum. Note that the maximum  $f(\alpha)$  value corresponds to  $D_0 = 0.537$  and that the end points  $\alpha_{min}$  and  $\alpha_{max}$  where  $f(\alpha) = 0$  correspond to correct values of  $D_\infty = 0.387$  and  $D_{-\infty} = 0.774$ , respectively, in Fig. 1(a).

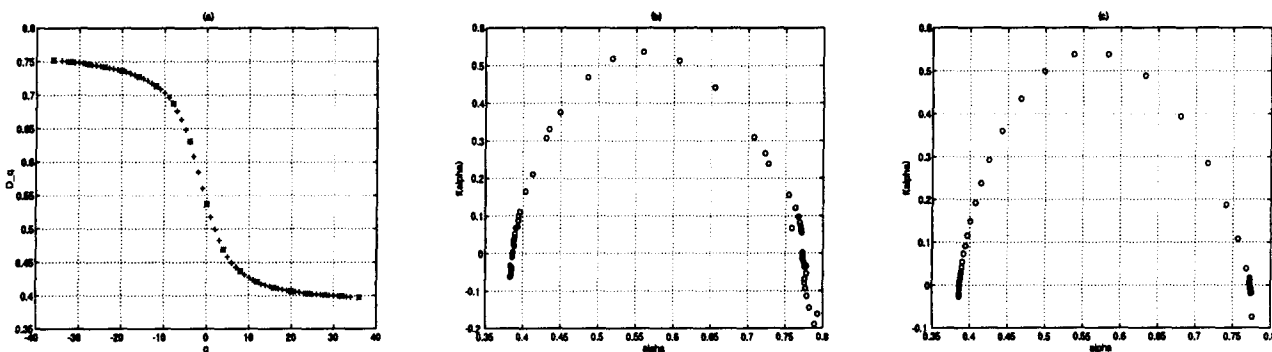


Figure 2: (a) Min-Max interpolated generalized dimensions of the Cantor set evaluated at 19 coarsely spaced  $D_q$  ( $\Delta q = 4$ ) values (\* = original sampled values, + = min-max interpolated values). (b)  $f(\alpha)$  computed using linearly interpolated  $D_q$  and centered first order difference equation. (c)  $f(\alpha)$  of the originally sampled  $D_q$  values in Fig. 2 (a) using min-max interpolator and differentiator. Notice the similarity between figures 1(b) and 2(c) except for the error below the  $f(\alpha) = 0$  line which are probably due to lowpass filter approximations.

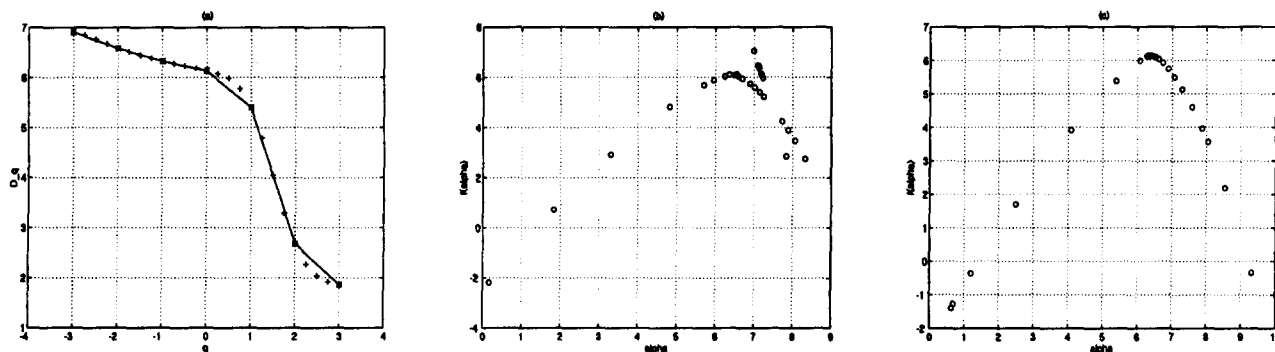


Figure 3: (a) Min-max interpolated GD,  $D_q$ , of the unvoiced fricative sound "S" spoken by an American Female Speaker (\* = original sampled values, + = min-max interpolated values, and solid line = linearly interpolated values). (b)  $f(\alpha)$  of linearly interpolated  $D_q$  in Fig. 3 (a) using centered first order difference equation. (c)  $f(\alpha)$  of unvoiced fricative using min-max interpolator/differentiator. The maximum value in the  $f(\alpha)$  curve corresponds to  $D_0 = 6.02$  and the estimated  $D_{-\infty}$  and  $D_\infty$  at  $f(\alpha) = 0$  are 9.3 and 1.5, respectively.