NUMERICAL INTEGRATION OF NONLINEAR MULTIDIMENSIONAL SYSTEMS

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ABSTRACT

The suitability of methods from multidimensional systems theory and digital signal processing for the efficient simulation of time and space dependent problems has already been demonstrated. Properly chosen functional transformations for the time and space coordinates turn a partial differential equation into a transfer function description of a multidimensional continuous system. It serves as the starting point for the derivation of a discrete system which closely models the behaviour of the given continuous system and which is suitable for computer implementation. This concept is extended here to the simulation of nonlinear multidimensional systems. The essence of the presented method is a systematic way to turn a nonlinear partial differential equation into a set of ordinary differential equations, for which standard methods for the numerical integration exist. This paper briefly reviews the linear case, points out the various difficulties arising from nonlinearity and shows how to overcome them. Numerical results demonstrate the effectiveness of the method.

1. INTRODUCTION

Problems which depend on continuous variables like time and space are generally modelled by differential equations. If the quantities in this model are considered as input and output signals, then such an idealized description is also called a continuous system. Purely time dependent problems are described by ordinary differential equations (ODE), leading to a onedimensional (or lumped parameter) system. Time and space dependent problems like wave propagation or heat and mass transfer are represented by partial differential equations (PDE), leading to multidimensional (or distributed parameter) systems. These systems are called nonlinear, if the coefficients of the PDE depend on the solution itself.

The conventional way to simulate the behaviour of multidimensional systems is to apply finite difference or finite element methods for the discretization of the time and space variables. This leads to large systems of equations, which have to be solved in each time step. Iterative methods are most popular for this task. However, the computational load is considerable, because a threefold loop is required in the case of nonlinear systems: For each time step, a number of iterations has to be performed, while each iteration step needs an inner loop to resolve the nonlinearity.

A recent alternative is the use of signal processing methods for the numerical simulation of multidimensional systems. One example are multidimensional wave-digital filters [2]. A different approach is based on the description of the continuous system by the generalized Fourier or Sturm–Liouville transformation (SLT) [1, 3]. The application of this transformation to the transfer function description of a continuous system and the derivation of the corresponding discrete simulation system is shown in [4, 6, 7] for the linear case. A first approach to nonlinear systems is given in [5].

We will extend here the functional transformation method for the simulation of multidimensional systems to an important case of nonlinear systems. To make the presentation more comprehensible, we will restrict the problem to one spatial dimension. At first, the linear case will be considered. Then the difficulties introduced by nonlinearities are discussed and a systematic procedure for turning a nonlinear partial differential equation into a set of ordinary differential equations is presented.

2. PROBLEM DEFINITION

As a simple example for a multidimensional system, we consider the PDE

$$\begin{aligned} \dot{y}(x,t) - \frac{1}{c}\left(\lambda y'(x,t)\right)' &= 0 \qquad x_0 \le x \le x_1 \\ y(x_0,t) &= y_b(t) \\ y(x_1,t) &= 0 \end{aligned} \tag{1}$$

It describes e.g. heat transfer or mass diffusion processes in one spatial dimension with material constants c and λ. The independent variables are space x and time t. Prime and dot denote spatial and temporal differentiation, respectively. The solution $y(x,t)$ is the time and space dependent potential, e.g. temperature or concentration. Without loss of generality, we set one of the boundary values $y(x_0,t)$ and $y(x_1,t)$ to zero.

The system is linear when the material constants c and λ are independent of y and nonlinear otherwise. The linear case has been treated also for more general boundary conditions, more spatial dimensions and other types of PDEs in [4, 6, 7]. A more general nonlinear problem definition is given in [5].

3. LINEAR SYSTEMS

The general idea of converting a PDE into a set of ODEs by applying functional transformations is best introduced for constant material parameters c and λ. The PDE in (1) is then linear

$$\dot{y}(x,t) - a y''(x,t) = 0 \qquad x_0 \leq x \leq x_1 \qquad (2)$$

with $a = \lambda/c$. From the methods presented in [4, 6, 7] follows, that a suitable generalized Fourier transformation

$$\mathcal{T}\{y(x,t)\} = \bar{y}(\beta_\mu, t) = \int_{x_0}^{x_1} y(x,t) K(x,\beta_\mu)\, dx \qquad (3)$$

takes here the form of a finite sine transformation with

$$K(x,\beta_\mu) = \sin(\beta_\mu(x-x_0)/\sqrt{a})\,,\ \beta_\mu = \frac{(\mu-1)\pi}{x_1-x_0}\sqrt{a}\,, \mu \in \mathbb{N} \qquad (4)$$

This special choice of the transformation kernel $K(x,\beta_\mu)$ yields the differentiation theorem

$$\mathcal{T}\{y''(x,t)\} = \frac{\beta_\mu}{\sqrt{a}} y_b(t) - \frac{\beta_\mu^2}{a}\bar{y}(\beta_\mu,t) \qquad (5)$$

which is easily proven by repeated integration by parts. Application of the transformation (3) and the differentiation theorem (5) to the PDE (2) gives the desired representation of the PDE by a set of ODEs, which can be written in the standard form for numerical integration

$$\dot{\bar{y}}(\beta_\mu,t) = -\beta_\mu^2 \bar{y}(\beta_\mu,t) + \sqrt{a}\beta_\mu y_b(t) = f(\bar{y}, y_b)\,. \qquad (6)$$

Equation (6) can be solved numerically at discrete times t_k for a finite number of spectral components β_μ. The solution of the PDE (2) can be recovered from the results $\bar{y}(\beta_\mu, t_k)$ of (6) by the inverse transformation $\mathcal{T}^{-1}$ to (3). Due to the discrete spectrum and the orthogonality of the kernel functions $K(x,\beta_\mu)$, $\mathcal{T}^{-1}$ takes the form of an orthogonal series expansion with respect to $K(x,\beta_\mu)$:

$$\mathcal{T}^{-1}\{\bar{y}(\beta_\mu,t)\} = y(x,t) = \sum_{\mu=1}^{\infty} \frac{1}{N_\mu}\bar{y}(\beta_\mu,t)K(x,\beta_\mu)\,, \qquad (7)$$

where N_μ is a normalization factor. This series needs to be evaluated only at the points in space x_n of interest. Unlike finite difference or finite element methods, no spatial grid refinement is necessary in order to increase the accuracy of the solution.

However, a potential drawback of performing the inverse transformation by evaluating a truncated series is the possibility of convergence problems. They arise when non-zero boundary values in (1) are approximated by a series of sine-functions (Gibbs phenomenon). These problems can be avoided by splitting the solution into two parts:

$$y(x,t) = u(x,t) + d(x,t) \qquad (8)$$

The first part $u(x,t)$ fulfills the boundary conditions in (1) and can be expressed in closed form as $u(x,t) = u_0(x)y_b(t)$. The second part $d(x,t)$ is zero at the boundaries and therefore possesses a rapidly convergent transform $\bar{d}(\beta_\mu,t) = \mathcal{T}\{d(x,t)\}$ (see [7]).

We can thus summarize the simulation procedure for a multidimensional system given by the PDE (2)

- Set up a system of ODEs by application of a suitable functional transformation to the PDE.
- Integrate the system of ODEs by numerical methods.
- Subtract a partial solution $\bar{u}(\beta_\mu,t) = \mathcal{T}\{u(x,t)\} = \bar{u}_0(\beta_\mu)y_b(t)$ to obtain a rapidly convergent series $\mathcal{T}^{-1}\{\bar{d}(\beta_\mu,t)\}$.
- Compute $d(x,t)$ numerically by truncated series expansion at the grid points (x_n, t_k).
- Add the partial solution $u(x,t) = u_0(x)y_b(t)$ and arrive at the result $y(x,t)$.

The structure of the system of ODEs is shown in figure 1.

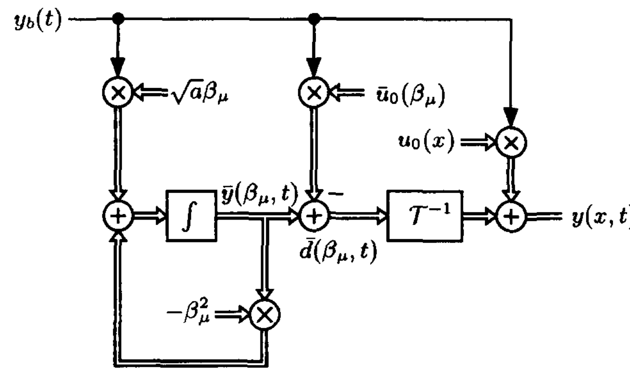

Figure 1: Structure of the system of ODEs representing the PDE (2)

4. NONLINEAR SYSTEMS

4.1. Problem Definition

Next, we consider the application of the functional transformation method outlined above to the nonlinear problem

$$\begin{aligned} \dot{y}(x,t) - \frac{1}{c(y)}\left(\lambda(y)y'(x,t)\right)' &= 0 \qquad x_0 \leq x \leq x_1 \\ y(x_0,t) &= y_b(t) \\ y(x_1,t) &= 0 \end{aligned} \qquad (9)$$

where the positive material constants $c(y) > 0$ and $\lambda(y) > 0$ depend on the solution $y(x,t)$. The nonlinearity thus introduced demands special attention to the following points:

- The nonlinear operator of spatial derivates in (9) has to be converted to a form more amenable to the application of functional transformations.
- Criteria for the choice of a suitable functional transformation have to be defined.
- Potential convergence problems for the inverse transformation have to be identified and considered.

These problems are addressed briefly in the sequel.

4.2. Conversion of the Nonlinear Operator

A transformation of the variable $y(x,t)$ of the form

$$w(x,t) = \mathcal{A}\{y(x,t)\} = \int_0^{y(x,t)} \lambda(\eta)\, d\eta \qquad (10)$$

converts the PDE in (9) into the familiar form (see (2))

$$\begin{aligned} \dot{w}(x,t) - \tilde{a}(w)w''(x,t) &= 0 \qquad x_0 \le x \le x_1 \\ w(x_0,t) &= w_b(t) = \mathcal{A}\{y_b(t)\} \\ w(x_1,t) &= 0 \end{aligned} \tag{11}$$

with $\tilde{a}(w) = a(y) = \lambda(y)/c(y)$. We now solve (11) for $w(x,t)$ and finally recover $y(x,t)$ by the inverse mapping $y = \mathcal{A}^{-1}\{w\}$.

4.3. Functional Transformation

Although the PDE (11) is of a very similar form as (11) for the linear case, it is not straightforward to find the transformation kernel of a functional transformation for $w(x,t)$

$$\mathcal{T}\{w(x,t)\} = \bar{w}(\beta_\mu,t) = \int_{x_0}^{x_1} w(x,t)K(x,\beta_\mu)\,dx\,, \tag{12}$$

since the coefficient $\tilde{a}(w)$ depends on the desired solution w. The transformation kernel $K(x,\beta_\mu)$ has to be chosen such that the transformation of the nonlinear term $\tilde{a}(w)w''(x,t)$ in (11) can be expressed by $\bar{w}(\beta_\mu,t)$ in a form similar to the differentiation theorem (5). In order to develop criteria for a proper choice of the transformation kernel, we apply (12) to $\tilde{a}(w)w''(x,t)$ and integrate by parts twice

$$\mathcal{T}\{\tilde{a}(w)w''(x,t)\} = \int_{x_0}^{x_1} \tilde{a}(w)w''(x,t)K(x,\beta_\mu)\,dx \tag{13}$$

$$= A_1 + A_2 + \int_{x_0}^{x_1} w(x,t)(\tilde{a}(w)K(x,\beta_\mu))''\,dx \tag{14}$$

with the abbreviations

$$A_1 = [w'(x,t)\tilde{a}(w) - w(x,t)\tilde{a}'(w)]K(x,\beta_\mu)\Big|_{x_0}^{x_1}\,, \tag{15}$$

$$A_2 = -w(x,t)\tilde{a}(w)K'(x,\beta_\mu)\Big|_{x_0}^{x_1} = w_b(t)\tilde{a}(w_b)K'(x_0,\beta_\mu) \tag{16}$$

The integral term in (14) could be turned into a transformation (12), if we were able to express $(\tilde{a}(w)K(x,\beta_\mu))''$ by a term proportional to $K(x,\beta_\mu)$. Due to the nonlinearity of the problem, any such expression would depend on w and consequently, no general solution is attainable.

To circumvent this problem, we split the nonlinear coefficient $\tilde{a}(w)$ into a constant part $\tilde{a}_0$ and into a part $\tilde{a}_1(w)$ which contains the dependence on w

$$\tilde{a}(w) = \tilde{a}_0 + \tilde{a}_1(w). \tag{17}$$

In many cases, such a composition is suggested by the problem at hand. For example, if $\tilde{a}(w)$ is obtained by a truncated Taylor expansion, then $\tilde{a}(w) = \tilde{a}_0 + \tilde{a}_1 \cdot w$, where $\tilde{a}_0$ and $\tilde{a}_1$ are constants. Application of (17) to (14) gives

$$\mathcal{T}\{\tilde{a}(w)w''(x,t)\} = A_1 + A_2 + A_3 + B(\beta_\mu,w) \tag{18}$$

with

$$A_3 = \int_{x_0}^{x_1} w(x,t)(\tilde{a}_0K''(x,\beta_\mu))\,dx\,, \tag{19}$$

$$B(\beta_\mu,w) = \int_{x_0}^{x_1} w(x,t)(\tilde{a}_1(w)K(x,\beta_\mu))''\,dx\,. \tag{20}$$

Now we observe, that (18) takes a much simpler form, if $K(x,\beta_\mu)$ is chosen as the solution of the Sturm–Liouville problem

$$\tilde{a}_0K''(x,\beta_\mu) = -\beta_\mu^2K(x,\beta_\mu)\,, \tag{21}$$

$$K(x_0,\beta_\mu) = 0\,, \tag{22}$$

$$K(x_1,\beta_\mu) = 0\,. \tag{23}$$

The solution of this homogeneous boundary value problem yields the same transformation kernel as for the linear case in (4). The effect on (18) is twofold: A_1 vanishes due to the homogeneous boundary conditions, and A_3 can be replaced by $-\beta_\mu^2\bar{w}(\beta_\mu,t)$. The result is an expression which is very similar to the differentiation theorem (5) for the linear case with a replaced by $\tilde{a}_0$.

$$\mathcal{T}\{\tilde{a}(w)w''(x,t)\} = -\beta_\mu^2\bar{w}(\beta_\mu,t) + w_b(t)\tilde{a}(w_b)K'(x_0,\beta_\mu) + B(\beta_\mu,w)\,. \tag{24}$$

At this point, we have established the criteria (21-23) for the choice of the transformation kernel and we have obtained an expression for the transformation of the nonlinear term $\tilde{a}(w)w''(x,t)$ in (11). Now we are ready to apply the transformation (12) to the PDE (11). The result is a set of ODEs

$$\dot{\bar{w}}(\beta_\mu,t) = -\beta_\mu^2\bar{w}(\beta_\mu,t) + \tilde{a}(w_b)\frac{\beta_\mu}{\sqrt{\tilde{a}_0}}w_b(t) + B(\beta_\mu,w)\,. \tag{25}$$

The structure of this system of equations is very similar to the linear case (compare (6)). The nonlinear coefficient $\tilde{a}(w)$ accounts for an additional term $B(\beta_\mu,w)$ which depends on $\tilde{a}_1(w)$ and cannot be resolved by a functional transformation. However a further analysis reveals, that $B(\beta_\mu,w)$ may also be expressed in terms of $\bar{w}(\beta_\mu,t)$. The result is a system of ODEs which differs from (6) in two aspects:

- it contains nonlinear differential equations,
- the equations are coupled, i.e. each single equation contains terms $\bar{w}(\beta_\mu,t)$ with different values for μ.

4.4. Convergence

Potential convergence problems are taken care of in the same way as for the linear problem by splitting the solution into two parts (see (8)). This separation has also to be considered in the treatment of the nonlinear term $B(\beta_\mu,w)$.

A slightly simplified version of the final structure is shown in fig. 2. It differs from fig. 1 only in the influence of the boundary value and in the presence of the coupling term $\bar{B}$ which is derived from $B(\beta_\mu,w)$ in (25).

5. NUMERICAL RESULTS

As an example, a heat flow problem in a solid with temperature dependent thermal diffusivity according to (9,11) was studied. A reference solution was calculated by a finite difference method (FDM) with small space and time step sizes. The boundary value $y_b(t)$ in (9) is defined piecewise by a ramp followed by a constant value (see fig. 3). The corresponding dynamical temperature variation starts

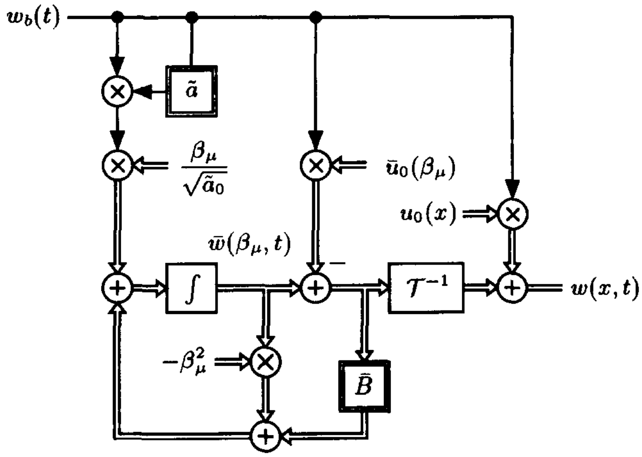

Figure 2: Simplified structure of the system of ODEs representing the nonlinear PDE (11)

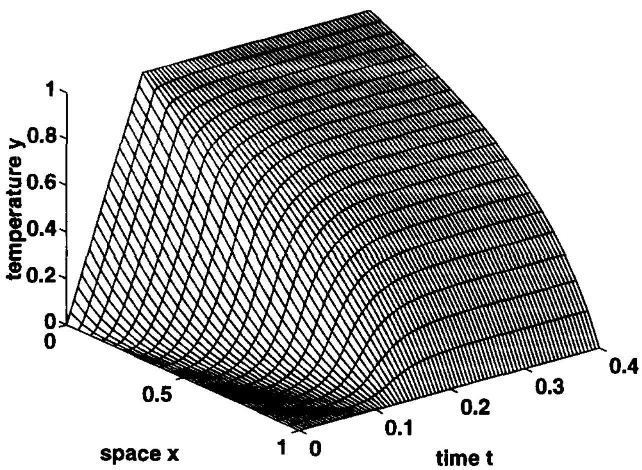

Figure 3: Reference solution of the example problem in normalized coordinates

at $t = 0$ from zero temperature and approaches a steady state solution, which shows a curved spatial dependence, thus displaying the nonlinear character of the problem.

The PDE was numerically solved at the grid points shown in fig. 3 with the FDM which was also used for the reference solution and with the functional transformation method (FTM) explained in this paper. Both methods allow to trade off accuracy against computing time: The FDM by decreasing the space and time step sizes and the FTM by increasing the number of eigenvalues β_μ involved in the series expansion of $\mathcal{T}^{-1}$ in (7). In both cases, the maximum of the deviation from the reference was recorded and plotted against the computing time necessary to achieve this approximate solution. The FDM achieves a higher accuracy on a finer grid (indicated by subsampling factor $r = 2$) only at the cost of extended computing time. The FTM gives the same accuracy at much less computational expense by a moderate increase in the number M of eigenvalues β_μ. These results were obtained with MATLAB on a SUN Sparc 20.

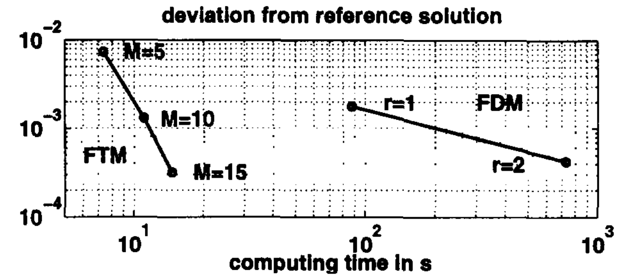

Figure 4: Comparison with Finite Difference Method

6. CONCLUSIONS

Transformation methods for the simulation of multidimensional linear systems can also be applied to nonlinear problems defined by partial differential equations. The key is the decomposition of the nonlinear operator into a linear and a nonlinear part. The linear part determines the kernel of a functional transformation, which allows to formulate the nonlinear problem as a set of ordinary differential equations (ODEs). The transformation plays an essential role in the calculation of the ODE coefficients and the determination of the solution in the original domain. The inverse transformation can be computed efficiently from a rapidly convergent series through a suitable separation of the solution.

7. REFERENCES

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