THIRD ORDER VOLTERRA SYSTEM IDENTIFICATION

Panos Koukoulas *Nicholas Kalouptsidis*
University of Athens
Department of Informatics
Division of Communications and Signal Processing
T.Y.P.A. Buildings, 157 71 Athens, Greece
koukoula@di.uoa.gr, kalou@di.uoa.gr

ABSTRACT

This paper is concerned with third order Volterra system identification. It is shown that crosscumulant information can be converted into a Fredholm integral equation. Closed form expressions for the Volterra kernels are derived using the determinant theory. Finally, special emphasis is focused on IID inputs.

1. INTRODUCTION

Identification of nonlinear Volterra systems is important in a wide range of applications. Closed form expressions for the determination of the Volterra kernels have been determined when the input is a zero mean stationary Gaussian process [1]. The Gaussian assumption is not always realistic. Katzenelson and Gould [2] described an iterative method to solve the continuous time problem. A discrete frequency domain approach for third order Volterra systems which leads to a system of linear equations solved with least squares approach, is described in [3]. The special case of zero mean IID inputs is treated in [4] for third order Volterra systems. Whilst the approach of [2]- [4] leads to estimates of the Volterra kernels, no closed form expressions are derived. Closed form expressions for second order Volterra system identification using "crosscumulant" analysis was pursued in [5].

Let $x(k)$ be a stationary discrete time random process. The $p-th$ order cumulant of $x(k)$, is denoted $c_{px}(k_1,k_2,\cdots,k_{p-1})$. The $p-th$ order polyspectrum $C_{px}(\omega_1,\omega_2,\cdots,\omega_{p-1})$ is defined as the $(p-1)$-dimensional discrete Fourier transform of $c_{px}(k_1,k_2,\cdots,k_{p-1})$. Crosscumulant and cross-polyspectra of two jointly stationary stochastic processes are similarly defined.

The Volterra system of order 3 has the form

$$y(n) = h_0 + \sum_{k_1=0}^{\infty} h_1(k_1)u(n-k_1) + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} h_2(k_1,k_2)$$

$$\times u(n-k_1)u(n-k_2) + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} h_3(k_1,k_2,k_3)$$

$$\times u(n-k_1)u(n-k_2)u(n-k_3) + \eta(n) \qquad (1)$$

The input $u(n)$ and the disturbance $\eta(n)$ are zero mean independent stationary stochastic processes. The Volterra kernels $h_1(k_1)$, $h_2(k_1,k_2)$, $h_3(k_1,k_2,k_3)$ are causal, absolutely summable sequences. Furthermore $h_2(k_1,k_2)$ and $h_3(k_1,k_2,k_3)$ are symmetric sequences.

2. FREDHOLM INTEGRAL EQUATIONS

Using the basic properties of cumulants and Leonov - Shiryaev theorem [8] the crosscumulant of $y(n)$ with one, two and three copies of the input can be computed. The resulting expressions are subsequently converted into the frequency domain. The pertinent formulas are

$$h_0 = c_y - \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{2u}(w_4)H_2(w_4,-w_4)dw_4 - \left(\frac{1}{2\pi}\right)^2$$

$$\times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} C_{3u}(w_4,w_5)H_3(w_4,w_5,-w_4-w_5)dw_4dw_5 \qquad (2)$$

$$C_{yu}(-w) = H_1(w)C_{2u}(w) + \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{3u}(w-w_4,w_4)$$

$$\times H_2(w-w_4,w_4)dw_4 + \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} C_{4u}(w-w_4-w_5,w_4,w_5)$$

$$\times H_3(w-w_4-w_5,w_4,w_5)dw_4dw_5 + 3C_{2u}(w)$$

$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{2u}(w_4)H_3(w,-w_4,w_4)dw_4 \qquad (3)$$

$$C_{yuu}(-w_1,-w_2) = H_1(w_1+w_2)C_{3u}(-w_1,-w_2)+$$

$$+2H_2(w_1,w_2)C_{2u}(w_1)C_{2u}(w_2) + \frac{1}{2\pi}$$

$$\times \int_{-\pi}^{\pi} C_{4u}(-w_2,w_1+w_2-w_4,w_4)H_2(w_1+w_2-w_4,w_4)dw_4+$$

$$+\left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} C_{5u}(-w_2,w_1+w_2-w_4-w_5,w_4,w_5)$$

$$\times H_3(w_1+w_2-w_4-w_5,w_4,w_5)dw_4dw_5 + 3C_{3u}(-w_1,-w_2)$$

$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{2u}(w_4)H_3(w_1+w_2,-w_4,w_4)dw_4 + 3C_{2u}(w_1)$$

$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{3u}(w_2-w_4,w_4)H_3(w_1,w_2-w_4,w_4)dw_4 + 3C_{2u}(w_2)$$

$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{3u}(w_1-w_4,w_4)H_3(w_2,w_1-w_4,w_4)dw_4 \qquad (4)$$

$$C_{yuuu}(-w_1,-w_2,-w_3) = H_1(w_1+w_2+w_3)C_{4u}(-w_1,-w_2,$$

$$-w_3) + \frac{1}{2\pi}\int_{-\pi}^{\pi} C_{5u}(-w_2, -w_3, w_1 + w_2 + w_3 - w_4, w_4)$$

$$\times H_2(w_1 + w_2 + w_3 - w_4, w_4) dw_4 + 2H_2(w_1, w_2 + w_3) C_{2u}(w_1)$$

$$\times C_{3u}(-w_2, -w_3) + 2H_2(w_2, w_1 + w_3) C_{2u}(w_2) C_{3u}(-w_1, -w_3)$$

$$+2H_2(w_3, w_1 + w_2) C_{2u}(w_3) C_{3u}(-w_1, -w_2) + \left(\frac{1}{2\pi}\right)^2$$

$$\times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} C_{6u}(-w_2, -w_3, w_1 + w_2 + w_3 - w_4 - w_5, w_4, w_5)$$

$$\times H_3(w_1 + w_2 + w_3 - w_4 - w_5, w_4, w_5) dw_4 dw_5 + 3C_{4u}(-w_1,$$

$$-w_2, -w_3)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{2u}(w_4) H_3(w_1 + w_2 + w_3, -w_4, w_4) dw_4 +$$

$$+3C_{2u}(w_1)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{4u}(-w_3, w_2 + w_3 - w_4, w_4)$$

$$\times H_3(w_1, w_2 + w_3 - w_4, w_4) dw_4 + 3C_{2u}(w_2)\frac{1}{2\pi}\int_{-\pi}^{\pi}$$

$$\times C_{4u}(-w_3, w_1 + w_3 - w_4, w_4) H_3(w_2, w_1 + w_3 - w_4, w_4) dw_4 +$$

$$+3C_{2u}(w_3)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{4u}(-w_1, w_1 + w_2 - w_4, w_4)$$

$$\times H_3(w_3, w_1 + w_2 - w_4, w_4) dw_4 + 3C_{3u}(-w_2, -w_3)\frac{1}{2\pi}\int_{-\pi}^{\pi}$$

$$\times C_{3u}(w_1 - w_4, w_4) H_3(w_2 + w_3, w_1 - w_4, w_4) dw_4 + 3C_{3u}(-w_1,$$

$$-w_3)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{3u}(w_2 - w_4, w_4) H_3(w_1 + w_3, w_2 - w_4, w_4) dw_4 +$$

$$+3C_{3u}(-w_1, -w_2)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{3u}(w_3 - w_4, w_4) H_3(w_1 + w_2, w_3 - w_4,$$

$$w_4) dw_4 + 6H_3(w_1, w_2, w_3) C_{2u}(w_1) C_{2u}(w_2) C_{2u}(w_3) \qquad (5)$$

The kernels h_0 and $H_1(w)$ can be expressed explicitly in terms of $H_2(w_1, w_2)$ and $H_3(w_1, w_2, w_3)$ while $H_2(w_1, w_2)$ and $H_3(w_1, w_2, w_3)$ satisfy a system of Fredholm integral equations once we restrict on the line $w_1 + w_2 = w$ and on the plane $w_1 + w_2 + w_3 = w$, with w fixed but arbitrary.

Indeed eq. (2) expresses h_0 in terms of $H_2(w_1, w_2)$ and $H_3(w_1, w_2, w_3)$. If eq. (3) is solved with respect to $H_1(w)$ yields

$$H_1(w) = \frac{C_{yu}(-w)}{C_{2u}(w)} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{C_{3u}(w - w_4, w_4)}{C_{2u}(w)} H_2(w - w_4, w_4)$$

$$\times dw_4 - \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{C_{4u}(w - w_4 - w_5, w_4, w_5)}{C_{2u}(w)}$$

$$\times H_3(w - w_4 - w_5, w_4, w_5) dw_4 dw_5 -$$

$$-3\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{2u}(w_4) H_3(w, -w_4, w_4) dw_4 \qquad (6)$$

As long as we move on the line $w_1 + w_2 = w$, eq. (4), after substituting $H_1(w)$ from (6), takes the form

$$H_2(w - w_2, w_2) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{C_{4u}(-w_2, w - w_4, w_4)}{2C_{2u}(w - w_2)C_{2u}(w_2)} - \right.$$

$$\left. - \frac{C_{3u}(-(w - w_2), -w_2) C_{3u}(w - w_4, w_4)}{2C_{2u}(w) C_{2u}(w - w_2) C_{2u}(w_2)}\right) H_2(w - w_4, w_4)$$

$$dw_4 + \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\frac{C_{5u}(-w_2, w - w_4 - w_5, w_4, w_5)}{2C_{2u}(w - w_2) C_{2u}(w_2)} - \right.$$

$$\left. - \frac{C_{3u}(-(w - w_2), -w_2) C_{4u}(w - w_4 - w_5, w_4, w_5)}{2C_{2u}(w) C_{2u}(w - w_2) C_{2u}(w_2)}\right)$$

$$H_3(w - w_4 - w_5, w_4, w_5) dw_4 dw_5 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3C_{3u}(w_2 - w_4, w_4)}{2C_{2u}(w_2)}$$

$$H_3(w - w_2, w_2 - w_4, w_4) dw_4 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3C_{3u}(w - w_2 - w_5, w_5)}{2C_{2u}(w - w_2)}$$

$$\times H_3(w_2, w - w_2 - w_5, w_5) dw_5 = \frac{C_{yuu}(-(w - w_2), -w_2)}{2C_{2u}(w - w_2) C_{2u}(w_2)}$$

$$- \frac{C_{yu}(-w) C_{3u}(-(w - w_2), -w_2)}{2C_{2u}(w) C_{2u}(w - w_2) C_{2u}(w_2)} \qquad (7)$$

Likewise if we move on the plane $w_1 + w_2 + w_3 = w$, eq. (5), after substituting $H_1(w)$ from (6) and $H_2(w_1, w_2)$ from (7), takes the form

$$H_3(w - w_2 - w_3, w_2, w_3) + \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{A_1(w_2, w_3, w_4, w_5)}{A_2(w_2, w_3)}$$

$$\times H_3(w - w_4 - w_5, w_4, w_5) dw_4 dw_5 +$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{C_{4u}(-w_3, w_2 + w_3 - w_4, w_4)}{2C_{2u}(w_2) C_{2u}(w_3)} - \right.$$

$$\left. - \frac{C_{3u}(-w_2, -w_3) C_{3u}(w_2 + w_3 - w_4, w_4)}{2C_{2u}(w_2 + w_3) C_{2u}(w_2) C_{2u}(w_3)}\right)$$

$$\times H_3(w - w_2 - w_3, w_2 + w_3 - w_4, w_4) dw_4 +$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{C_{4u}(-w_3, w - w_2 - w_5, w_5)}{2C_{2u}(w - w_2 - w_3) C_{2u}(w_3)} - \right.$$

$$\left. - \frac{C_{3u}(-(w - w_2 - w_3), -w_3) C_{3u}(w - w_2 - w_5, w_5)}{2C_{2u}(w - w_2) C_{2u}(w - w_2 - w_3) C_{2u}(w_3)}\right)$$

$$\times H_3(w_2, w - w_2 - w_5, w_5) dw_5 +$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{C_{4u}(-w_2, w - w_3 - w_4, w_4)}{2C_{2u}(w - w_2 - w_3) C_{2u}(w_2)} - \right.$$

$$\left. - \frac{C_{3u}(-(w - w_2 - w_3), -w_2) C_{3u}(w - w_3 - w_4, w_4)}{2C_{2u}(w - w_3) C_{2u}(w - w_2 - w_3) C_{2u}(w_2)}\right)$$

$$\times H_3(w_3, w - w_3 - w_4, w_4) dw_4 +$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{A_3(w_2, w_3, w_4)}{A_2(w_2, w_3)} H_2(w - w_4, w_4) dw_4 =$$

$$= f_2(w_2, w_3) \qquad (8)$$

where A_1, A_2, A_3 are expressions involving input cumulants of order at most six [6]. Likewise $f_2(w_2, w_3)$ involves crosscumulants of y and at most 3 copies of u. Equations (7) and (8) form a system of two 2-dimensional Fredholm integral equations of the second kind of the form

$$x_1(w_2, w_3) - \lambda\int_a^b\int_a^b K_{11}(w_2, w_3, w_4, w_5) x_1(w_4, w_5) dw_4 dw_5 -$$

$$-\lambda\int_a^b\int_a^b K_{12}(w_2, w_3, w_4, w_5) x_2(w_4, w_5) dw_4 dw_5 =$$

$$= f_1(w_2, w_3) \quad (9)$$

$$x_2(w_2, w_3) - \lambda \int_a^b \int_a^b K_{21}(w_2, w_3, w_4, w_5) x_1(w_4, w_5) dw_4 dw_5 -$$

$$-\lambda \int_a^b \int_a^b K_{22}(w_2, w_3, w_4, w_5) x_2(w_4, w_5) dw_4 dw_5 =$$

$$= f_2(w_2, w_3) \quad (10)$$

The system of Fredholm equations (9), (10) can be converted into a single two dimensional Fredholm equation via proper concatenation. This is done as follows. Define the kernel

$$K(w_2, w_3, w_4, w_5) = \begin{cases} K_{11}(w_2, w_3, w_4, w_5) \\ K_{21}(w_2 - 2\pi, w_3 - 2\pi, w_4, w_5) \\ K_{12}(w_2, w_3, w_4 - 2\pi, w_5 - 2\pi) \\ K_{22}(w_2 - 2\pi, w_3 - 2\pi, w_4 - 2\pi, w_5 - 2\pi) \end{cases}$$

Moreover, let

$$f(w_2, w_3) = \begin{cases} f_1(w_2, w_3), & -\pi \le w_2, w_3 < \pi \\ f_2(w_2 - 2\pi, w_3 - 2\pi), & \pi \le w_2, w_3 \le 3\pi \end{cases}$$

and

$$x(w_2, w_3) = \begin{cases} x_1(w_2, w_3), & -\pi \le w_2, w_3 < \pi \\ x_2(w_2 - 2\pi, w_3 - 2\pi), & \pi \le w_2, w_3 \le 3\pi \end{cases}$$

Then the system of integral equations (9), (10) is transformed into the single integral equation

$$x(w_2, w_3) - \lambda \int_{-\pi}^{3\pi} \int_{-\pi}^{3\pi} K(w_2, w_3, w_4, w_5) x(w_4, w_5) dw_4 dw_5$$

$$= f(w_2, w_3) \quad (11)$$

This is a two dimensional Fredholm equation of the second kind. The determinant theory of multidimensional Fredholm equations is similar to the one dimensional case [7]. The coresponding Fredholm determinant is

$$\Delta(\lambda) = 1 + \sum_{\nu=1}^{\infty} (-1)^\nu \frac{\lambda^\nu}{\nu!} \int_{-\pi}^{3\pi} \cdots \int_{-\pi}^{3\pi}$$

$$\times K \begin{pmatrix} \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \cdots & \xi_{\nu 1}\xi_{\nu 2} \\ \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \cdots & \xi_{\nu 1}\xi_{\nu 2} \end{pmatrix} d\xi_{11} d\xi_{12} \cdots d\xi_{\nu 1} d\xi_{\nu 2}$$

If $\Delta(\lambda)$ is nonzero, (11) has a unique continuous solution given by

$$x(w_2, w_3) = f(w_2, w_3) -$$

$$-\lambda \int_{-\pi}^{3\pi} \Gamma(w_2, w_3, w_4, w_5; \lambda) f(w_4, w_5) dw_4 dw_5$$

where

$$\Gamma(w_2, w_3, w_4, w_5; \lambda) = -\frac{\Delta(w_2, w_3, w_4, w_5; \lambda)}{\Delta(\lambda)}$$

and

$$\Delta(w_2, w_3, w_4, w_5; \lambda) = K(w_2, w_3, w_4, w_5) + \sum_{\nu=1}^{\infty} (-1)^\nu \frac{\lambda^\nu}{\nu!}$$

$$\times \int_{-\pi}^{3\pi} \cdots \int_{-\pi}^{3\pi} K \begin{pmatrix} w_2 w_3 & \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \cdots & \xi_{\nu 1}\xi_{\nu 2} \\ w_4 w_5 & \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \cdots & \xi_{\nu 1}\xi_{\nu 2} \end{pmatrix}$$

$$\times d\xi_{11} d\xi_{12} \cdots d\xi_{\nu 1} d\xi_{\nu 2}$$

The notation

$$K \begin{pmatrix} \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \cdots & \xi_{\nu 1}\xi_{\nu 2} \\ \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \cdots & \xi_{\nu 1}\xi_{\nu 2} \end{pmatrix} \equiv$$

$$\begin{vmatrix} K(\xi_{11}, \xi_{12}, \xi_{11}, \xi_{12}) & \cdots & K(\xi_{11}, \xi_{12}, \xi_{\nu 1}, \xi_{\nu 2}) \\ K(\xi_{21}, \xi_{22}, \xi_{11}, \xi_{12}) & \cdots & K(\xi_{21}, \xi_{21}, \xi_{\nu 1}, \xi_{\nu 2}) \\ \vdots & \ddots & \vdots \\ K(\xi_{\nu 1}, \xi_{\nu 2}, \xi_{11}, \xi_{12}) & \cdots & K(\xi_{\nu 1}, \xi_{\nu 2}, \xi_{\nu 1}, \xi_{\nu 2}) \end{vmatrix}$$

is employed.

The above closed form expressions are difficult to compute. Simpler situations result if the input is suitably restricted. The next section discusses IID inputs.

3. THIRD ORDER VOLTERRA IDENTIFICATION WITH IID INPUT

Suppose that the input signal, $u(n)$, is an IID zero mean random process with

$$c_{ku}(i_1, \cdots, i_{k-1}) = \gamma_k \delta(i_1, \cdots, i_{k-1})$$

Suppose further that the following persistent excitation conditions hold: $\Pi_1 \neq 0$, $\Pi_2 \neq 0$, and $\Pi_3 \neq 0$, where $\Pi_1 = \gamma_2$, $\Pi_2 = \gamma_4\gamma_2 - \gamma_3^2 + 2\gamma_2^3$ and

$$\Pi_3 = \gamma_6\gamma_4\gamma_2 - \gamma_6\gamma_3^2 + 2\gamma_6\gamma_2^3 - \gamma_5^2\gamma_2 + 2\gamma_5\gamma_4\gamma_3 - 12\gamma_5\gamma_3\gamma_2^2 -$$

$$-\gamma_4^3 + 12\gamma_4\gamma_3^2\gamma_2 + 7\gamma_4^2\gamma_2^2 + 24\gamma_4\gamma_2^4 - 9\gamma_3^4 - 24\gamma_3^2\gamma_2^3 + 12\gamma_2^6$$

Then the Volterra kernels are given by

$$h_0 = c_y + \frac{\gamma_5\gamma_2^2 - 2\gamma_4\gamma_3\gamma_2 + \gamma_3^3 + 4\gamma_3\gamma_2^3}{\Pi_3} \left(\frac{1}{2\pi}\right)^2$$

$$\times \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} C_{yuuu}(w_4 + w_5, -w_4, -w_5) dw_4 dw_5 -$$

$$-\frac{\gamma_6\gamma_2^2 - \gamma_5\gamma_3\gamma_2 - \gamma_4^2\gamma_2 + \gamma_4\gamma_3^2 + 9\gamma_4\gamma_2^3 + 3\gamma_3^2\gamma_2^2 + 6\gamma_2^5}{\Pi_3}$$

$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{yuu}(w_4, -w_4) dw_4 + \left(\frac{\gamma_6\gamma_3\gamma_2 - \gamma_5\gamma_4\gamma_2 - \gamma_5\gamma_3^2}{\Pi_3} + \right.$$

$$\left. + \frac{\gamma_4^2\gamma_3 + 5\gamma_4\gamma_3\gamma_2^2 + 3\gamma_3^3\gamma_2 + 6\gamma_3\gamma_2^4}{\Pi_3} \right) C_{yu}(0) \quad (12)$$

$$H_1(w) = \frac{\gamma_2\Pi_3 + \Pi_2(\gamma_5\gamma_3 - \gamma_4^2 - 5\gamma_4\gamma_2^2 + 9\gamma_3^2\gamma_2 - 6\gamma_2^4)}{\Pi_2\Pi_3}$$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} C_{yuuu}(-(w - w_4 - w_5), -w_4, -w_5) dw_4 dw_5 -$$

$$-\frac{\gamma_2}{\Pi_2} \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{yuuu}(-w, w_4, -w_4) dw_4 + \frac{\gamma_3}{\Pi_2} C_{yuu}(-w, 0) -$$

$$-\left(\frac{\gamma_6\gamma_3 - \gamma_5\gamma_4 - 3\gamma_5\gamma_2^2 + 6\gamma_4\gamma_3\gamma_2 + 9\gamma_3^3 - 12\gamma_3\gamma_2^3}{\Pi_3} + \right.$$

$$\left. + \frac{\gamma_3}{\Pi_2} \right) \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{yuu}(-(w - w_4), -w_4) dw_4 +$$

$$+ \left(\frac{\gamma_6\gamma_4 + 2\gamma_6\gamma_2^2 - \gamma_5^2 - 15\gamma_5\gamma_3\gamma_2 + 12\gamma_4^2\gamma_2 9\gamma_4\gamma_3^2}{\Pi_3} + \right.$$

$$+\frac{30\gamma_4\gamma_2^3 - 36\gamma_3^2\gamma_2^2 + 12\gamma_2^5}{\Pi_3}\Bigg) C_{yu}(-w) \quad (13)$$

$$H_2(w_1, w_2) = \frac{\gamma_3\Pi_3 - \gamma_2\Pi_2(\gamma_5\gamma_2 - \gamma_4\gamma_3 + 6\gamma_3\gamma_2^2)}{\gamma_2\Pi_2\Pi_3}\left(\frac{1}{2\pi}\right)^2$$

$$\times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} C_{yuuu}(-(w_1 + w_2 - w_4 - w_5), -w_4, -w_5)dw_4 dw_5 -$$

$$-\frac{\gamma_3}{2\gamma_2\Pi_2}\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuuu}(-w_1, -(w_2 - w_4), -w_4)dw_4 -$$

$$-\frac{\gamma_3}{2\gamma_2\Pi_2}\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuuu}(-w_2, -(w_1 - w_4), -w_4)dw_4 +$$

$$+\Bigg(\frac{\gamma_6\gamma_2 - \gamma_4^2 + 9\gamma_4\gamma_2^2 + 9\gamma_3^2\gamma_2 + 6\gamma_2^4}{\Pi_3} -$$

$$-\frac{\gamma_4\gamma_2 + \gamma_3^2 + 2\gamma_2^3}{2\gamma_2^2\Pi_2}\Bigg)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuu}(-(w_1 + w_2 - w_4), -w_4)dw_4 +$$

$$+\frac{\gamma_4\gamma_2 + \gamma_3^2 + 2\gamma_2^3}{2\gamma_2^2\Pi_2} C_{yuu}(-w_1, -w_2) -$$

$$-\frac{\gamma_6\gamma_3 - \gamma_5\gamma_4 + 3\gamma_4\gamma_3\gamma_2 + 9\gamma_3^3 + 6\gamma_3\gamma_2^3}{\Pi_3} C_{yu}(-w_1 - w_2) \quad (14)$$

$$H_3(w_1, w_2, w_3) = \frac{C_{yuuu}(-w_1, -w_2, -w_3)}{6\gamma_2^3} +$$

$$+\frac{(\gamma_4\gamma_2 - \gamma_3^2 - \gamma_2^3)\Pi_3 + 3\gamma_2^3\Pi_2^2}{3\gamma_2^3\Pi_2\Pi_3}\left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}$$

$$\times C_{yuuu}(-(w_1 + w_2 + w_3 - w_4 - w_5), -w_4, -w_5)dw_4 dw_5 -$$

$$-\frac{\gamma_4\gamma_2 - \gamma_3^2}{6\gamma_2^3\Pi_2}\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuuu}(-w_1, -(w_2 + w_3 - w_4), -w_4)dw_4 -$$

$$-\frac{\gamma_4\gamma_2 - \gamma_3^2}{6\gamma_2^3\Pi_2}\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuuu}(-w_2, -(w_1 + w_3 - w_4), -w_4)dw_4 -$$

$$-\frac{\gamma_4\gamma_2 - \gamma_3^2}{6\gamma_2^3\Pi_2}\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuuu}(-w_3, -(w_2 + w_3 - w_4), -w_4)dw_4 -$$

$$-\frac{\gamma_3}{3\gamma_2\Pi_2} C_{yuu}(-w_1, -w_2 - w_3) - \frac{\gamma_3}{3\gamma_2\Pi_2} C_{yuu}(-w_2, -w_1 - w_3) -$$

$$-\frac{\gamma_3}{3\gamma_2\Pi_2} C_{yuu}(-w_3, -w_1 - w_2) - \Bigg(\frac{\gamma_5\gamma_2 - \gamma_4\gamma_3 + 6\gamma_3\gamma_2^2}{\Pi_3} -$$

$$-\frac{\gamma_3}{\gamma_2\Pi_2}\Bigg)\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{yuu}(-(w_1 + w_2 + w_3 - w_4), -w_4)dw_4 +$$

$$+\frac{\gamma_5\gamma_3 - \gamma_4^2 - 2\gamma_4\gamma_2^2 + 6\gamma_3^2\gamma_2}{\Pi_3} C_{yu}(-w_1 - w_2 - w_3) \quad (15)$$

A detailed account is given in [6].

REFERENCES

[1] P. Koukoulas and N. Kalouptsidis, "Nonlinear System Identification Using Gaussian Inputs", *IEEE Trans. on Signal Processing*, vol. SP-43, pp. 1831-1841, August 1995.

[2] J. Katzenelson and L. A. Gould, "The Design of Nonlinear Filters and Control Systems. Part I", *Information and Control*, 5, pp. 108-143, 1962.

[3] S. W. Nam and E. J. Powers, "Application of Higher Order Spectral Analysis to Cubically Nonlinear System Identification", *IEEE Trans. on Signal Processing*, vol. SP-42, pp. 1746-1765, July 1994.

[4] C. H. Tseng and E. J. Powers, "Identification of Cubic Systems Using Higher Order Moments of I.I.D. Signals", *IEEE Trans. on Signal Processing*, vol. SP-43, pp. 1733-1735, July 1995.

[5] P. Koukoulas and N. Kalouptsidis, "Second Order Volterra System Identification", *8th IEEE SP Workshop SSAP*, Corfu, Greece, pp. 383-386, 1996.

[6] P. Koukoulas and N. Kalouptsidis, "Second and Third Order Volterra System Identification", *IEEE Trans. on Information Theory*, under review.

[7] J. Kondo, *Integral Equations*, Oxford University Press, 1991.

[8] V. P. Leonov and A. N. Shiryaev, "On a Method of Calculation of Semi-Invariants", *Theory Prob. Appl.*, vol.4, pp. 319-329, 1959.