OPTIMUM LOW COST TWO CHANNEL IIR ORTHONORMAL FILTER BANK †

Jamal Tuqan and P. P. Vaidyanathan

Department of Electrical Engineering 136-93
California Institute of Technology
Pasadena, CA 91125
tuqan@systems.caltech.edu, ppvnath@sys.caltech.edu

ABSTRACT

In this paper, we statistically optimize a well known class of IIR two channel orthonormal filter banks parameterized by a single coefficient when subband quantizers are present. The optimization procedure is extremely simple and very fast compared for example to the linear programming method used in the FIR case to achieve similar compaction (coding) gains. The special form of the filters assure the existence of a zero at π which can be important for some wavelet applications and eliminate some of the major concerns that arise in the FIR design case. Finally, the compaction gain obtained is high and numerically very close to two (ideal case) for low pass spectra, high pass spectra and certain cases of multiband spectrum. For these cases, the use of higher order IIR filters does not increase the compaction (coding) gain.

1. INTRODUCTION

There has been a considerable interest in designing filter banks while taking into account the effect of subband quantization [1], [2], [3]. Given a fixed budget of b bits for the subband quantizers, the goal is to simultaneously optimize the analysis and synthesis filters and to choose a subband bit allocation strategy such that the average variance of the error $e(n)$ at the output of the subband coder is minimized.

The energy compaction problem. Consider the scheme of Fig. 1.1 which shows a filter $H(e^{j\omega})$ with input $x(n)$ and output $y(n)$. The input $x(n)$ is assumed to be a widesense stationary process. With the input power spectral density $S_{xx}(e^{j\omega})$ fixed, the compaction filter problem is to find $H(e^{j\omega})$ such that the variance of the output, given by

$$\sigma_y^2 = \int_{-\pi}^{\pi} S_{xx}(e^{j\omega})|H(e^{j\omega})|^2 \frac{d\omega}{2\pi} \qquad (1)$$

is maximized under the constraint that $|H(e^{j\omega})|^2$ is a Nyquist(M) filter. The design of such compaction filters is important in its own merit because of the usual question of finding the best wavelets for a given application. For a two channel orthonormal filter bank, the two problems are identical : *optimizing one of the subband filters for maximum energy compaction is equivalent to designing a two channel optimal orthonormal subband coder.* To see this, recall that the coding gain [4] of an orthonormal filter bank under optimum bit allocation and with the high bit rate quantizer assumptions is given by $\mathcal{G}_{SBC}(2) = \frac{\sigma_x^2}{\sqrt{\sigma_{x_0}^2 \sigma_{x_1}^2}}$ where σ_x^2 is the input signal variance and $\sigma_{x_k}^2$ is the variance of the kth subband signal. Using $2\sigma_x^2 = \sigma_{x_0}^2 + \sigma_{x_1}^2$, the above expression can be rewritten as

$$\mathcal{G}_{SBC}(2) = \frac{1}{\sqrt{G_{comp}(2)(2 - G_{comp}(2))}} \qquad (2)$$

where $G_{comp}(2)$ is the so called **compaction gain** and is equal to $\sigma_{x_0}^2/\sigma_x^2$. The compaction gain therefore uniquely determines the coding gain of a 2-channel orthonormal subband coder. It is important to keep in mind that *the maximum possible compaction gain $G_{comp}(2)$ is equal to two whereas the coding gain $\mathcal{G}_{SBC}(2)$ can be arbitrarily large.*

2. THE PROBLEM SET UP

The aim of this paper is to statistically optimize a two channel orthonormal filter bank when subband quantizers are present at low cost. Two channel orthonormal filter banks are of special interest because they form a basic building block in the design of wavelet transforms. Low cost filters are quite attractive in image and audio coding applications. The requirement for a very efficient two channel system motivates the investigation of filter banks based on IIR filters rather than FIR ones. Moreover, previous work on finite order compaction filters and/or finite order two channel optimum orthonormal filter banks has been mainly dedicated to the FIR case (see for example [5], [6], and [7] to name a few). To meet the above requirements, we propose the optimization of a class of two channel IIR orthonormal filter banks based on the sum of two all

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pass filters [8]. In specific, consider the single coefficient system:

$$H_0(z) = \frac{1}{\sqrt{2}}(z^{-2} + z^{-1}\frac{z^{-2}-\alpha}{1-\alpha z^{-2}}) \ , \ H_1(z) = H_0(-z) \tag{1}$$

where α is real and $-1 < \alpha < 1$. The synthesis filters are given by $F_0(e^{j\omega}) = H_0^*(e^{j\omega})$ and $F_1(e^{j\omega}) = H_1^*(e^{j\omega})$. The two channel system is shown in Fig. 2.1. Note that since the polyphase components of the analysis filters are stable causal all pass filters, their reciprocals will produce unstable synthesis filters. To overcome this difficulty, Ramstad [9] proposed to implement the inverse filters as anti-causal stable IIR filters. Although (1) is a seemingly restrictive case, the proposed form of the filter $H_0(z)$ is a special case of the more general structure introduced recently by Phoong and al. [10]. It has been shown that this type of filter provides several excellent advantages [10]. For example, the filter $H_0(z)$ (and therefore $H_1(z)$) can have a very good frequency response. Furthermore, the special form of the filter assure the existence of a zero at π which can be important for wavelet applications. For the purpose of this paper, our results indicate that for the cases where we can obtain high compaction gain with the special filters in (1), using higher order filters does not increase the compaction gain.

3. THE ANALYTICAL RESULTS

Consider the set up shown in Fig. 2.1. where the input signal $x(n)$ is a zero mean wide-sense stationary (WSS) random process with a power spectrum $S_{xx}(e^{j\omega})$. Each subband quantizer, labeled by $\mathcal{Q}$, represents a scalar uniform (PCM) quantizer and is modeled as an additive zero mean white noise source $q(n)$ with variance

$$\sigma_q^2 = c2^{-2b}\sigma_{x_i}^2 \tag{1}$$

where σ_q^2 is the quantization noise variance, c is a constant that depends on the statistical distribution of the subband signal $x_i(n)$ and the overflow probability, and $\sigma_{x_i}^2$ is the variance of the *ith* subband signal. The subband coding problem reduces to finding the optimum coefficient α_{opt} that maximizes the compaction gain (alternatively the subband variance) at the output of one of the subband. The specific form of the analysis filters given in (1) guarantee automatically the Nyquist property and transforms the constrained optimization problem into an unconstrained one. A closed form expression for the compaction gain is given next.

Proposition 1. *Consider the scheme of Fig. 2.1 under all the previous filter and quantizer assumptions. The compaction gain at the output of one of the subband filters, say $H_0(z)$, can be expressed as follows :*

$$\begin{aligned} G_{comp}(2) = 1 &+ \frac{R_{xx}(1)}{R_{xx}(0)} \\ &- (\alpha+\alpha^2)\frac{R_{xx}(1)}{R_{xx}(0)} \\ &+ \frac{(\alpha-\alpha^3)}{R_{xx}(0))}\sum_{n=0}^{\infty}\alpha^n R_{xx}(2n+3) \end{aligned} \tag{2}$$

The proof can be found in [11]. The infinite summation in (2) is the result of the IIR nature of the filter bank. The above equation was written specifically in the above form in order to emphasize the following points: First, when α is equal to zero, the compaction gain is equal to $1 + R_{xx}(1)/R_{xx}(0)$, which is simply the 2×2 KLT compaction gain. This indeed makes sense since the structure of Fig. 2.1 reduces to the 2×2 universal KLT. Second, when the input signal is white noise, i.e. $R_{xx}(k) = \delta(k)$, the compaction gain is equal to one. Finally, observe that the above equation involves only the odd samples of the autocorrelation sequence $R_{xx}(k)$, due to the Nyquist constraint on $|H_0(e^{j\omega})|^2$. Therefore, if the input signal $x(n)$ is such that its power spectrum $S_{xx}(z)$ takes the form $S(z^2)$, the compaction gain is equal to unity.

The goal now is to find the optimum coefficient α_{opt} that maximizes (2). In general, it is difficult to obtain analytical solutions due to the complexity of the expression in (2). We will therefore present analytical solutions for the optimum coefficient α_{opt}, the compaction gain $G_{comp}(2)$ and the coding gain $\mathcal{G}_{SBC}(2)$ only for specific examples of the input $x(n)$ such as the MA(1) and AR(1) processes. For a general random process $x(n)$, the optimum coefficient α_{opt} is obtained numerically through a MATLAB program.

Example 3.1. *Case of a MA(1) process.* Assume that the input $x(n)$ is a zero mean MA(1) process with an autocorrelation sequence in the form

$$R_{xx}(k) = \begin{cases} 1 & k = 0. \\ \dfrac{\theta}{1+\theta^2} & k = 1, -1. \\ 0 & \text{otherwise.} \end{cases}$$

where θ is between -1 and 1. It can be shown [11] that, for this case,

$$\begin{aligned} &\alpha_{opt} = -0.5 \\ &G_{comp}(2) = 1 + 5R_{xx}(1)/4R_{xx}(0) \\ &\mathcal{G}_{SBC}(2) = 1/\sqrt{1 - 25R_{xx}^2(1)/16R_{xx}^2(0)} \end{aligned} \tag{3}$$

It is interesting to note that the optimum coefficient α_{opt} is independent of the signal statistics.

Example 3.2. *Case of an AR(1) process.* Assume now that the input $x(n)$ is a zero mean AR(1) process with an autocorrelation sequence in the form $R_{xx}(k) =$

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$\rho^{|k|}$ where ρ is between -1 and 1. It can be shown [11] that, for this case,

$$\alpha_{opt} = (1 - \sqrt{(1+\rho^2)})/\rho^2$$
$$G_{comp}(2) = 1 + \rho + \rho(1-\rho^2)\alpha_{opt}^2 \quad (4)$$
$$\mathcal{G}_{SBC}(2) = 1/\sqrt{1 - \rho^2(1+\alpha_{opt}^2(1-\rho^2))^2}$$

We note that the optimum coefficient in this case is independent of the sign of ρ, is always negative and between $1-\sqrt{2}$ and -0.5 (the case where $\rho = 0$). Furthermore, one can show that there is a negligible loss of compaction gain even when α_{opt} is implemented using very small number of binary shift and add operations [11]. As ρ approaches unity, the scheme is asymptotically equivalent to the 2×2 universal KLT.

4. EXAMPLES FOR MORE GENERAL INPUTS

We give several examples where the optimum coefficient α_{opt} is computed numerically through a MATLAB program. The program uses the compaction gain expression derived previously with input $R_{xx}(k)$ and output α_{opt}. Although written in MATLAB, the program converges in fractions of a second. This is an order of magnitude faster than linear programming techniques used to design high order FIR compaction filters to achieve similar compaction gain. We vary the input $x(n)$ such that the power spectral density shape spans a variety of choices from "low pass" to multiband with energy concentrated in a specific region to multiband with more even energy distribution. The magnitude squared response of the optimum IIR compaction filter together with the ideal optimum compaction filter magnitude squared response and the input power spectral density are shown in Fig. 4.1 to Fig 4.4. We adopt the following convention for all the plots : the **solid curve** denotes the input power spectral density, the **dash-dot** curve denotes the magnitude squared response of the optimum IIR compaction filter and the **dashed curve** represents the magnitude response of the ideal optimum compaction filter. This last curve is obtained by optimizing an FIR filter with order equal to 65 using a linear programming approach.

5. CONCLUDING REMARKS

We have addressed in this paper the problem of optimizing a two channel orthonormal filter bank with a cost constraint in mind. To achieve this, we optimized a well known class of IIR two channel orthonormal filter banks parameterized by a single coefficient. The resulting optimum filter bank provides some good advantages that are not available in the FIR case. First, the Nyquist property is satisfied automatically because of the special form of the filters. Second, in FIR compaction filter design, one would traditionally find $|H(e^{j\omega})|^2$ and then perform a spectral factorization to obtain $H(e^{j\omega})$. The positivity of the solution is not usually guaranteed and spectral factorization can be problematic in case of unit-circle zeros. *In our scheme, these concerns do not exist* since $H_0(z)$ is directly found. Third, the form of the filters assure the existence of a zero at π which can be important for some wavelet applications. Fourth, the filters have only one coefficient which can be quantized without a major sacrifice in compaction gain as we demonstrate in [11]. Finally, the compaction gain obtained is high and very close to two (ideal case) for low pass spectrums, high pass spectrums and certain cases of multiband spectrum. The only weakness of the filter bank under consideration is its poor performance for the case of general multiband spectrums. This is mainly due to the monotone property of the phase of an all pass function. For such cases, α_{opt} can be set to zero to obtain the 2×2 universal KLT.

References

[1] Kovacevic, J., "Subband coding systems incorporating quantizer Models", IEEE Trans. on Image Processing, pp. 543-553, May 1995.

[2] Uzun, N. and Haddad, R.A. , "Cyclostationary Modeling, analysis and optimal compensation of quantization errors in subband codecs", IEEE Trans. on SP, pp. 2109-2119, September 1995.

[3] Gosse, K. and Duhamel, P., "Perfect reconstruction versus MMSE filter banks in source coding", Submitted to the IEEE Trans. on SP.

[4] Jayant, N.S. and Noll, P. *Digital coding of waveforms,* Prentice Hall, Inc., Englewood Cliffs, 1984.

[5] Moulin, P.,"A new look at signal adapted QMF design", Proc. ICASSP, pp. 1312-1315, May 1995.

[6] Caglar, H., Liu, Y. and Akansu, A.,"Statistically optimized PR-QMF", SPIE vol. 1605, pp. 86-94, 1991.

[7] Kirac, A. and Vaidyanathan, P. P., "Theory and design of optimum FIR compaction filters", submitted to the SP special issue on filters banks and wavelets.

[8] Vaidyanathan, P.P. *Multirate systems and filter banks,* Prentice Hall, Inc., Englewood Cliffs, 1993.

[9] Ramstad, T.A., "IIR filterbank for subband coding of images", Proc. ISCAS, pp. 827-830, June 1988.

[10] Phoong, S.M., Kim, C.W., Vaidyanathan, P.P. and Ansari, R., "A new class of two channel biorthogonal filter banks and wavelet bases", IEEE Trans. on SP, pp. 649-665, March 1995.

[11] Tuqan, J. and Vaidyanathan, P. P., "Low cost optimum IIR compaction filters and their application in image compression", in preparation.

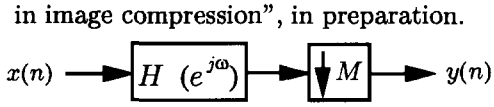

Fig. 1.1. Schematic of the energy compaction problem.

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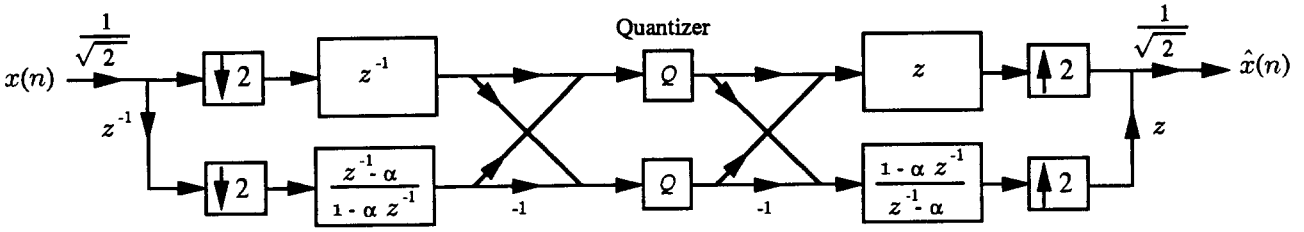

Fig. 2.1. The class of two channel orthonormal filter bank under consideration.

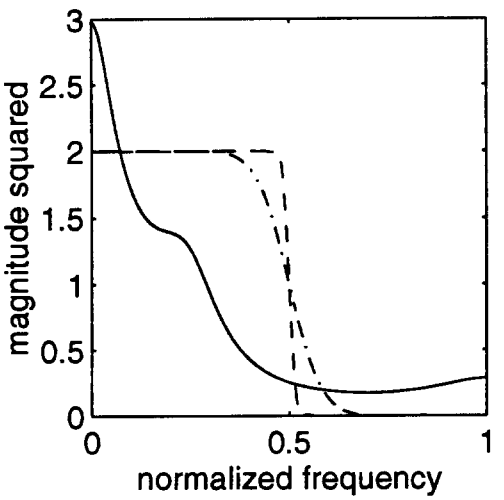

Fig. 4.1. Case of a low pass AR(5) process : IIR compaction gain = 1.95, theoretical compaction gain = 1.951 and IIR FB coding gain = 5.00 db.

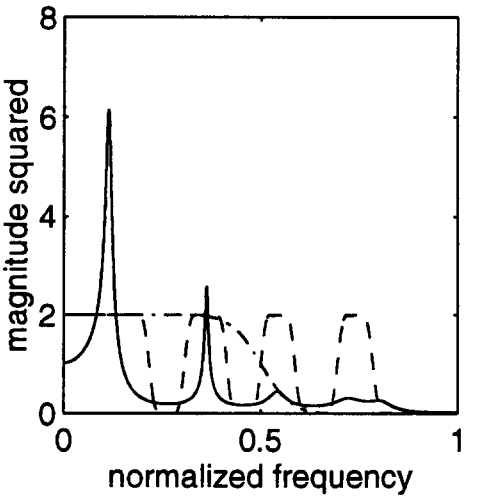

Fig. 4.2. Case of a multiband AR(12) process : IIR compaction gain = 1.951, theoretical compaction gain = 1.97 and IIR FB coding gain = 5.14 db.

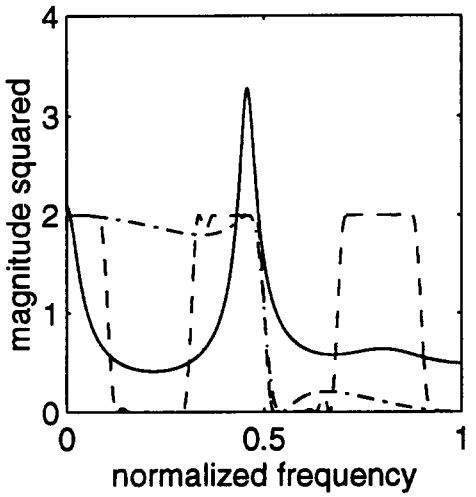

Fig. 4.3. Case of a multiband AR(10) process : IIR compaction gain = 1.52, theoretical compaction gain = 1.61 and IIR FB coding gain = 0.68 db.

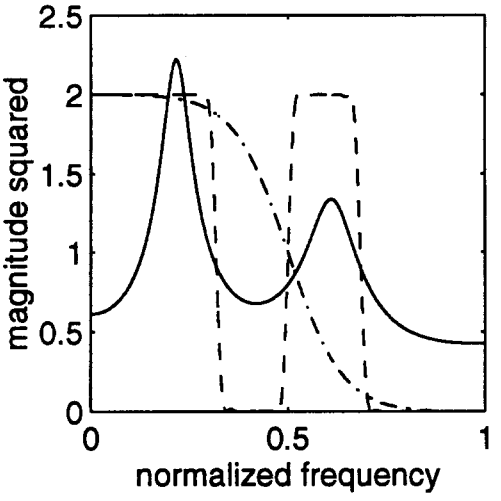

Fig. 4.4. Case of a multiband AR(5) process : IIR compaction gain = 1.387, theoretical compaction gain = 1.6 and IIR FB coding gain = 0.35 db.

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