DIRECT DESIGN OF NONUNIFORM FILTER BANKS

Takayuki Nagai   *Takaaki Futie*   *Masaaki Ikehara*

Department of Electrical Engineering, Keio University
3-14-1, Hiyoshi, Kohoku-ku, Yokohama, 223 JAPAN
nagai@tkhm.elec.keio.ac.jp

ABSTRACT

In this research, we propose a direct design method of nonuniform filter banks (NUFBs). This method is based on frequency domain constraints to eliminate the amplitude and the aliasing distortions. Both NUFBs with integer and rational sampling factors can be designed with common procedure. Here, we also consider the design method which requires only to solve linear equations iteratively. In our proposed method, least square error of the perfect reconstruction (PR) constraints is minimized without using the nonlinear programing technique.

1. INTRODUCTION

Multirate filter banks have been well studied in recent years and are used in various applications[1]. Especially, the design methods of uniform filter banks have been well established. In some applications such as audio and speech coding, however, the nonuniform division is desirable[2]. NUFBs have been considerd to meet this requirment[3].

The indirect method is known to be able to design PR NUFBs for some cases. This method, however, is not optimal as pointed out in [5]. The design methods based on modulation were reported in [6]-[8]. These methods are efficient but not exact reconstruction. The design method of PR NUFBs was presented in [4] for biorthogonal case. In [5], paraunitary NUFB was constructed using two transforms. However, the NUFB with integer factor cannot be designed because of a uniform filter bank containing the shifted versions of the same filter.

In this research, we propose a direct method to design PR NUFBs with rational and integer sampling factor. This method is based on frequency domain constraints to eliminate the amplitude and the aliasing distortions. So we can design PR NUFB directly and the same procedure is applicable to both cases of integer and rational sampling factor. We first formulate the least square error of the PR constraints in quadratic forms of filter coefficients vetors of both analysis and synthesis filters. The cost function is defined as this quadratic function. Then the minimization method is derived. Since the cost function is formulated in a quadratic form, one can iteratively solve for individual variables while fixing the other one. The overall error can be shown to be monotonously decreasing and optimal solution can be found. By usig this method, we can obatain LP biorthogonal NUFB without using nonlinear optimiza-
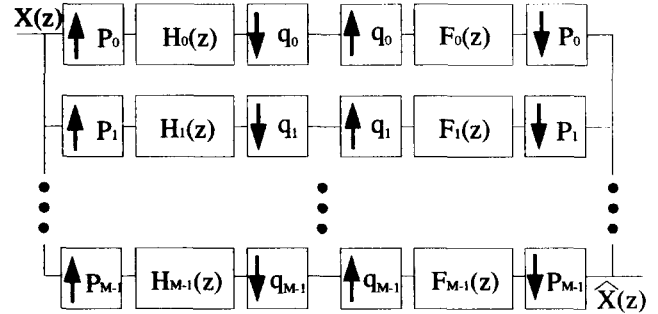

Figure 1. M-channel nonuniform filter bank

tion. This method is also applicable to the LP paraunitary NUFB design with some modifications. We also investigate the nessesary conditions for linear phase PR NUFBs.

2. NONUNIFORM FILTER BANKS

Fig.1 shows a NUFB with rational sampling factor. We assume that the p_i and q_i are coprime and the NUFB is critically sampled, that is

$$\sum_{i=0}^{M-1} \frac{p_i}{q_i} = 1. \tag{1}$$

Note that the case that all $p_i, (i = 0, 1, \cdots, M-1)$ equale to 1 is called NUFB with integer sampling factor.

Let the $H_k(z)$ and $F_k(z)$ be the analysis and the synthesis filters respectivly. The input signal $X(z)$ and the output signal $\hat{X}(z)$ are related as

$$
\begin{aligned}
\hat{X}(z) &= \sum_{i=0}^{M-1} \frac{1}{p_i q_i} \sum_{k=0}^{p_i-1} \sum_{l=0}^{q_i-1} H_i(z^{1/p_i} W_{q_i}^l W_{p_i}^k) \\
&\times F_i(z^{1/p_i} W_{p_i}^k) X(z W_{q_i}^{p_i l}),
\end{aligned} \tag{2}
$$

where $W_i = e^{-j2\pi/i}$. Thus the condition for elimination of the amplitude distortion coresponds to the term of $l = 0$ and is expressed by

$$\sum_{i=0}^{M-1} \frac{1}{p_i q_i} \sum_{k=0}^{p_i-1} H_i(z^{1/p_i} W_{p_i}^k) F_i(z^{1/p_i} W_{p_i}^k) = z^{-d}, \tag{3}$$

where d is some positive integer. The condition for the aliasing cancellation can be written as

$$\sum_{i=0}^{M-1} \frac{r_i}{p_i q} \sum_{k=0}^{p_i-1} \delta\left(\left\lfloor \frac{l}{r_i} \right\rfloor - \frac{l}{r_i}\right) H_i(z^{1/p_i} W_{q_i}^{(\frac{l+q\gamma}{p_i r_i})} W_{p_i}^k)$$

$$\times F_i(z^{1/p_i} W_{p_i}^k) = 0, \quad for \quad l = 1, \cdots, q-1, \quad (4)$$

where q is a least common multiple of q_i and $r_i = q/q_i$ ($i = 0, 1, \cdots, M-1$). γ denosts a minimum integer which makes $(l+q\gamma)/(p_i r_i)$ integer ($\gamma \geq 0$). Eq.(3) and Eq.(4) are the conditions for PR. In the next section we consider the design method, i.e., how to minimize Eq.(3) and Eq.(4).

3. DESIGN OF LP PR NUFB

In this section, we present a new design method for biorthogonal LP NUFB. We first formulate the PR condition as a quadratic form of the filter coefficients vectors. Then, the minimization procedure will be derived.

3.1. Quadratic formlation of the PR constraints

After this, we consider only the NUFB with integer sampling factor for simplisity. It is clear that the same procedure is applicable to the rational sampling factor case.

In our proposed method, the least square error of the PR condition expressed by Eq.(3) and Eq.(4) are minimized. A cost function Φ is defined as

$$\Phi = \Phi_a + \Phi_b, \quad (5)$$

$$\Phi_a = \alpha \int_0^\pi \left\{ \sum_{i=0}^{M-1} \frac{1}{q_i} H_i^*(e^{j\omega}) F_i^*(e^{j\omega}) - 1 \right\}^2 d\omega, \quad (6)$$

$$\Phi_b =$$
$$\beta \sum_{l=1}^{q-1} \int_0^\pi \left\{ \sum_{i=0}^{M-1} \frac{1}{q_i} \delta\left(\left\lfloor \frac{l}{r_i} \right\rfloor - \frac{l}{r_i}\right) H_i^*(e^{j\omega} W_q^l) F_i^*(e^{j\omega}) \right\}^2 d\omega, \quad (7)$$

where $H_k^*(e^{j\omega})$, $F_k^*(e^{j\omega})$ represent zero frequency responses as

$$H_k^*(e^{j\omega}) = s_k^T(\omega) h_k, \quad F_k^*(e^{j\omega}) = c_k^T(\omega) f_k,$$

where h_k, f_k represent the coefficients vectors of k-th analysis and synthesis filters, respectively and $c_i(\omega)$, $s_i(\omega)$ are appropriate trigonometrical vectors. α and β are weights for amplitude distortion and aliasing distortion, respectively.

Now, define the coefficients vector as follows,

$$h = \begin{bmatrix} h_0^T & h_1^T & \cdots & h_{M-1}^T \end{bmatrix}^T,$$
$$f = \begin{bmatrix} f_0^T & f_1^T & \cdots & f_{M-1}^T \end{bmatrix}^T.$$

The first term of Eq.(5) can be rewritten using the matrix form as,

$$\Phi_a = \alpha \int_0^\pi \left\{ h^T Q_a f - 1 \right\}^2 d\omega, \quad (8)$$

where

$$Q_a = block \quad diag \begin{bmatrix} \frac{1}{q_0} s_0(\omega) c_0^T(\omega) & \frac{1}{q_1} s_1(\omega) c_1^T(\omega) \\ \cdots & \frac{1}{q_{M-1}} s_{M-1}(\omega) c_{M-1}^T(\omega) \end{bmatrix}.$$

Moreover, Eq.(8) can be expanded as follows,

$$\Phi_a = h^T R_h h - 2p_h h + \alpha\pi$$
$$= f^T R_f f - 2p_f f + \alpha\pi, \quad (9)$$

where

$$R_h = \alpha \int_0^\pi Q_a f f^T Q_a^T d\omega \quad R_f = \alpha \int_0^\pi Q_a^T h h^T Q_a d\omega$$
$$p_h = \alpha \int_0^\pi f^T Q_a^T d\omega \quad p_f = \alpha \int_0^\pi h^T Q_a d\omega.$$

The aliasing error Φ_b can be expressed in a similar way,

$$\Phi_b = \beta \sum_{l=1}^{q-1} \int_0^\pi \left[h^T Q_l f f^T Q_l^T h \right] d\omega = h^T T_h h = f^T T_f f, \quad (10)$$

where

$$Q_l = block \quad diag \left[\frac{\delta\left(\left\lfloor \frac{l}{r_0} \right\rfloor - \frac{l}{r_0}\right)}{q_0} s_0\left(\omega - \frac{2\pi l}{q}\right) c_0^T(\omega) \right.$$

$$\left. \cdots \frac{\delta\left(\left\lfloor \frac{l}{r_{M-1}} \right\rfloor - \frac{l}{r_{M-1}}\right)}{q_{M-1}} s_{M-1}\left(\omega - \frac{2\pi l}{q}\right) c_{M-1}^T(\omega) \right],$$

$$T_h = \beta \sum_{l=1}^{q-1} \int_0^\pi Q_l f f^T Q_l^T d\omega,$$

$$T_f = \beta \sum_{l=1}^{q-1} \int_0^\pi Q_l^T h h^T Q_l d\omega.$$

The overall cost function $\Phi = \Phi_a + \Phi_b$ can be written as follows,

$$\Phi = h^T \Psi h - 2p_h^T h + \alpha\pi = f^T \Theta f - 2p_f^T f + \alpha\pi, \quad (11)$$

where

$$\Psi = R_h + T_h \quad and \quad \Theta = R_f + T_f,$$

which is in quadratic form of coefficient vectors.

3.2. Design algorithm of LP biorthogonal NUFB

The design procedure is as follows;

1. Design analysis filters using any suitable method such as LMS, remez algorithm and so on.

2. Fix the coefficients of analysis filters h and then, calculate synthesis filters f from Eq.(11)

3. Fix the coefficients of synthesis filters f to the value obtained by the above equation. Then, calculate analysis filters h in the same way.

4. Terminate if the cost function Φ is small enough($\Phi \leq 10^{-10}$), otherwise continue the algorithm (back to step 2)

The above algorithm yields LP biorthogonal NUFB and it requires only solving linear equation iteratively. Convergence of the algorithm is guaranteed and cost function can be shown to be monotonously decreasing.
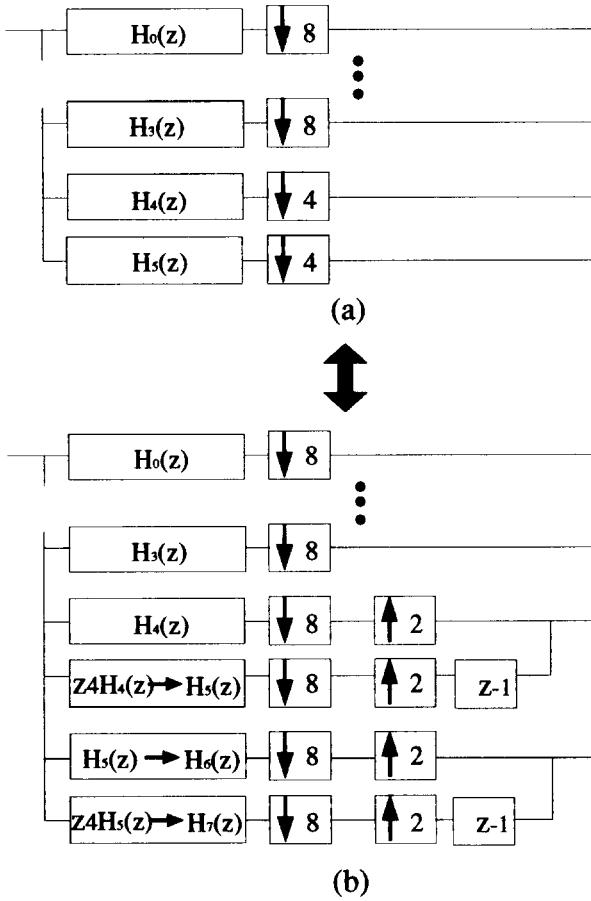
Figure 2. (a) **Analysis system.** (b) **Transformed (transform 2 in [5]) analysis system.**

3.3. Design of LP Paraunitary NUFB

The main difference between paraunitary and biorthogonal filter bank is in the choice of synthesis system. In paraunitary case, synthesis system is time reversed version of the analysis bank. In this case, one can use the same design algorithm as in the previous subsection. The only difference is that the coefficient vector is chosen to be the average of $h_k(n)$ and $f_k(n)$ in each iteration. In other words, the average value of analysis and synthesis filters is used for h_k instead of solving it in the step 3.

$$
\begin{aligned}
h_k^{(i+1)} &= (h_k^{(i)} + f_k^{(i)})/2 \quad \text{for symmetric} \\
h_k^{(i+1)} &= (h_k^{(i)} - f_k^{(i)})/2 \quad \text{for antisymmetric.}
\end{aligned}
$$

4. NESSESARY CONDITIONS FOR LINEAR PHASE

In this section, we consider nessesary conditions for linear phase. In uniform case, such conditons have reported in [9], however, it is difficult to derive them for all possible case of NUFBs. We consider here only the case that intger sampling factor, q is even and all q_i are $q/2$ or q. For example, [1/8 1/8 1/8 1/8 1/4 1/4] comes under this classification. As shown in [5], this system can be transformed into q-channel uniform FB as Fig.2. Therefore we

can expect that the nessesary conditions can be derived in a smilar manner.

Let the length of the i-th filter be $L_i = K_i q + \beta$, where β, K_i are positive integer and $0 \leq \beta < M$, $K_i \geq 1$. The polyphase matrix of Fig.2 $E(z)$ must satisfy the following property for linear phase,

$$E(z) = DZ(z)E(z^{-1})J(z) \qquad (12)$$

where,

$$
J(z) = \begin{bmatrix} z^{-1}J_\beta & 0_{\beta \times (q-\beta)} \\ 0_{(q-\beta) \times \beta} & J_{q-\beta} \end{bmatrix},
$$

$$
Z(z) = diag\left[z^{-(K_0 - 1 - \frac{2a_0}{q})} \cdots z^{-(K_{q-1} - 1 - \frac{2a_{q-1}}{q})} \right],
$$

and D is a diagonal matrix whose entry is 1 when the corresponding filter is symmetric and -1 when it is antisymmetric. a_i denotes the highest positive order of the i-th filter. By taking the determinant of both sides of Eq.(12), we get

$$
\begin{aligned}
|E(z)| &= |D|z^{-(\sum_{i=0}^{q-1} K_i) + q + \frac{2}{q}\sum_{i=0}^{q-1} a_i} \\
&\times |E(z^{-1})|(-1)^{(\frac{q}{2}+\beta)}z^{-\beta}. \qquad (13)
\end{aligned}
$$

When $z = 1$, following must hold,

$$|D|(-1)^{(\frac{q}{2}+\beta)} = 1. \qquad (14)$$

Since this equation is same as in [9], we can conclude that there are $q/2$ symmetric and $q/2$ antisymmetric filters if β is even and there are $q/2 + 1$ symmetric and $q/2 - 1$ antisymmetric filters if β is odd.

Evaluating Eq.(13) at $z = -1$ gives

$$(-1)^{-(\sum_{i=0}^{q-1} K_i) + q + \frac{2}{q}\sum_{i=0}^{q-1} a_i} = 1 \quad for\ even\ \beta, \qquad (15)$$

$$(-1)^{-(\sum_{i=0}^{q-1} K_i) + q + \frac{2}{q}\sum_{i=0}^{q-1} a_i} = -1 \quad for\ odd\ \beta. \qquad (16)$$

By noting that $\frac{2}{q}\sum_{i=0}^{q-1} a_i$ is even when M is even or odd when M is odd, we can get following results;

1. If β and M are even or β and M are odd, $\sum_{i=0}^{q-1} K_i$ must be even.

2. If β is odd and M is even or β is even and M is odd, $\sum_{i=0}^{q-1} K_i$ must be odd.

We must choose the length and symmetry acording as above conditions. We can also derive the conditions for rest of integer sampling factor case in a similar way, however, those of rational sampling factor case is open problem.

5. EXAMPLE

A 6-channel LP paraunitary NUFB with integer sampling factor was designed using our proposed method. The decimation ratio is [1/8 1/8 1/8 1/8 1/4 1/4] and each filter has length 12. The symmetry of each filter was chosen as [S A S A S A]. It is clear that these specifications satisfy the conditions for linear phase described in above section. Fig.3 and Fig.4 show a frequency response of the designed system and coefficients of each filter, respectively. The error vs the number of iterations is appeared in Fig.4. It can be seen that the error decreases monotonously.
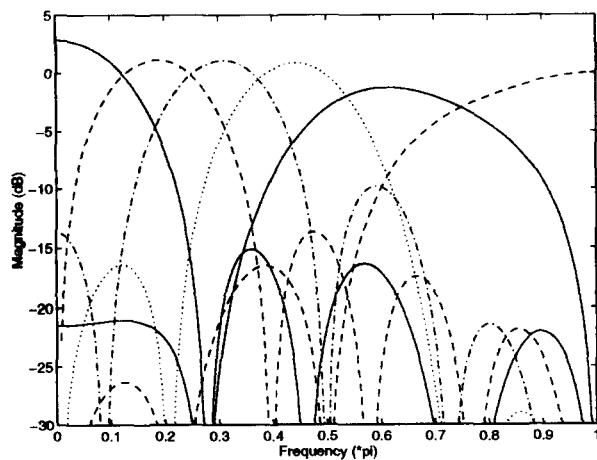
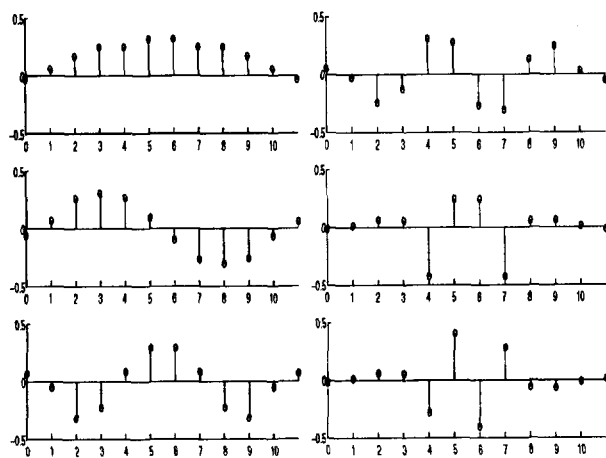**Figure 3. Magnitude response plots for designed system**

Figure 5. Square error of PR vs number of iteration

Figure 4. Coefficients of each filter

6. CONCLUSION

In this paper, we proposed a new approach to the design of PR NUFBs with linear phase. Since our proposed method minimizes the amplitude and the aliasing distortion in frequency domain directly, one can design both NUFBs with integer and rational sampling factors by the same porcedure. Besides, instead of applying nonlinear optimization, the proposed method only solves a set of linear equations iteratively. The iteration minimizes the PR constraints instead of imposing PR as a design constraint. As a result, more efficient method is obtained compared to any other design methods. We also investigated the nesessary conditions for linear phase PR NUFBs.

REFERENCES

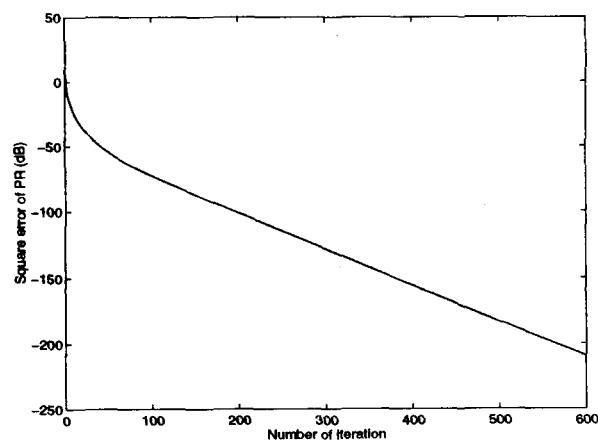[1] P. P. Vaidyanathan, "Multirate Systems and Filter Banks", Prentice-Hall,Englewood Cliffs,NJ,1993.

[2] R.N.J.Veldhuis, M.Breeuwer and R.G.Van der Waal, "Subband Coding of Digital Audio Signals", Philips J.Res., vol.44, NO.2/3, pp329-343, 1989.

[3] P.-Q.Hoang and P.P.Vaidyanathan, "Nonuniform Multirate Filter Banks:Theory and Design",in Proc. IEEE Int. Symp. Circuits Syst., Portland, OR, pp371-374, 1989.

[4] K. Nayebi, T. P. Barnwell III and M. J. T. Smith, "Nonuniform Filter Banks : A Reconstruction and Design Theory", IEEE Trans. Signal Processing, vol.41, NO.3, pp1114-1127, March 1993.

[5] J.Kovacevic and M.Vetterli, "Perfect Reconstruction Filter Banks with Rational Sampling Facters", IEEE Trans. Signal Processing, vol.41, NO.6, pp2047-2066, June 1993.

[6] Jeong-Jin Lee and Byeong Gi Lee, "A Design of Nonuniform Cosine Modulated Filter Banks", IEEE Trans. Circuits and Systems-II: Analog and Digital Signal processing, vol.42, NO.11, pp732-737, Nov.1995.

[7] John Princen, "The Design of Nonuniform Modulated Filter Banks", IEEE Trans. Signal Processing, vol.43, NO.11, pp2550-2560, Nov.1995.

[8] F.Argenti and E. Del Re, "Non-uniform Filter Banks Based on A Multi-prototype Cosine Modulation", in Proc. IEEE Int. Conf. Acoust. Speech & Signal Processing, Atlanta, 1996.

[9] Trac D. Tran and T. Q. Nguyen, "On M-channel Linear-Phase FIR Filter Banks and Application in Image Compression", preprint