RECONSTRUCTION FOR NOVEL SAMPLING STRUCTURES

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ABSTRACT

We examine the problem of reconstructing a signal from periodic non-uniform samples, *i.e.* a uniform train from which samples are deleted in some periodic fashion. We develop a condition previously derived by Herley and Wong and examine its implications.

We show that this method has a number of advantages over alternative approaches. In particular it gives a condition for achieving the minimum rate rather than approaching it asymptotically. We show that it generally leads to a reconstruction scheme that is simpler than those derived by other strategies. We examine a few special cases in which the minimum rate is precisely achieveable, and cases where design of the reconstruction system is possible without explicitly knowing the signal spectrum.

1. INTRODUCTION

The problem addressed in [4] is to find some strategy that will allow us to sample multiband signals, $s(t)$, at the minimum rate. By multiband we mean that the set of frequencies, $\mathcal{P}$, over which $S(f)$ is non-zero is a finite union of arbitrary non-overlapping open intervals:

$$\mathcal{P} = \bigcup_{i=0}^{L-1} \{(a_i, b_i) \cup (-b_i, -a_i)\}. \tag{1}$$

It is easily seen that for a multiband signal such as shown in Figure 1 it may not be possible to uniformly sample such that there are no gaps and no overlaps in the sampled spectrum. Instead we investigate periodic non-uniform sampling. This can be thought of as uniform sampling where samples are deleted in some periodic fashion. Periodic non-uniform approaches have previously been studied in [3, 5, 7].

We assume that $\mathcal{P}$ has some maximum frequency $f_{max} = b_{L-1}$, and the effective bandwidth of $S(f)$ is

$$B_{eff} = 2 \cdot \sum_{i=0}^{L-1} (b_i - a_i).$$

We know that uniform sampling at the rate $2f_{max}$ or higher is always possible, but this entails waste of bandwidth since $2f_{max} > B_{eff}$. If we choose some integer M, and frequency f_0 then sampling at Mf_0 is sufficient provided $Mf_0 \geq 2f_{max}$. Denote by $x(n)$ the discrete sequence which equals the rate-Mf_0 sampled signal at the sample points:

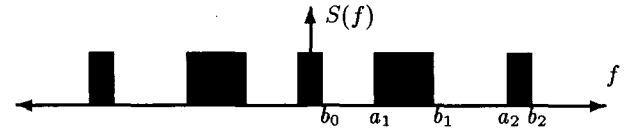$$x(n) = s(t)|_{t=n/(Mf_0)}. \tag{2}$$

Figure 1: Spectrum of a multiband signal. Periodic non-uniform sampling allows sampling at the minimum rate.

Now split the discrete sequence $x(n)$ into its M-phase components

$$X(z) = X_0(z^M) + z^{-1}X_1(z^M) + \cdots z^{-(M-1)}X_{M-1}(z^M).$$

Suppose it were possible to reconstruct $x(n)$ exactly using only N of the M components (*i.e.* reconstruct $X(z)$ from $X_i(z)$ where $i \in \mathcal{A}$ and $\mathcal{A}$ is a set that contains only N of the indices $\{0, 1, \cdots, M-1\}$). Since we can in turn reconstruct $s(t)$ from $x(n)$ this would mean that we could recover $s(t)$ from an average of Nf_0 samples per unit time. If in addition we could have

$$Nf_0 = B_{eff} \tag{3}$$

then the average number of samples required to reconstruct the analog signal $s(t)$ would be the same as the effective bandwidth, which is known to be the minimum rate [6].

The conditions to reconstruct a discrete-time sequence from N of its M-phase components were derived by Foster and Herley in [3]. We refer the interested reader to [3, 4] for details and reproduce here two of the important results.

Theorem 1.1 *If no more than N terms in the sum $\sum_{k=0}^{M-1} X(e^{j\omega + 2\pi k/M})$ are non-zero at any frequency, then $X(e^{j\omega})$ can be exactly reconstructed from some set of N of its M-phase components.*

Theorem 1.1 says that when $X(e^{j\omega})$ is subsampled by M, to reconstruct from N components, there should be no more than $N-1$ overlaps in the subsampled spectrum. Recall that $x(n)$ was derived from the rate-Mf_0 sampled signal. When this condition is satisfied we can sample $s(t)$ at rate Nf_0. In addition, to have minimum rate, *every* frequency should be hit by *precisely* N copies of the aliased spectrum [4]. In words: the condition for minimum rate uniform sampling was "no overlaps and no gaps," for periodic non-uniform sampling it becomes "$N-1$ overlaps and no gaps."

Thus when $Mf_0 \geq 2f_{max}$ we can recover $s(t)$ from the sequence $x(n)$. Further, when the requirement of Theorem 1.1 is satisfied (no frequency hit by more than N copies) we can recover $x(n)$ from only N of its M-phase components.

When in addition (3) holds (*every* frequency hit by precisely N copies) we can have minimum rate sampling. If we can find an f_0 that satisfies the requirement of Theorem 1.1, we can then choose any M that satisfies $M f_0 \geq 2 f_{max}$. It is not yet obvious whether finding such an f_0 is possible in general. The following Theorem helps [4].

Theorem 1.2 *To satisfy the requirement of Theorem 1.1 it is necessary and sufficient that*

$$\sum_{i=0}^{L-1} [e(g + b_i) - e(g + a_i) - e(g - b_i) + e(g - a_i)] = 0 \quad \forall \ g,$$

(4)

where

$$e(f) \triangleq \sum_{n \neq 0} \frac{(-1)^{n+1} f_0}{2 \pi j n} e^{j 2 \pi f n / f_0}.$$

(5)

Note that $e(f)$ is the Fourier series expansion of the periodic piecewise linear function shown in Figure 2. Of course it is a function of f_0. Observe from the figure, or by substituting into (5), that for any α and β whose difference is and integer times f_0, that $e(\alpha) = e(\beta)$. This allows us to see more clearly how solutions to (4) may be found. For example, suppose that we had $b_i - a_i = k_i f_0$ for some integer $k_i, \ \forall \ i$. This would give $e(g + b_i) = e(g + a_i), \ \forall \ g$, and then (4), would be satisfied.

2. MINIMUM RATE SAMPLING

Let us summarise the foregoing, and paraphrase the results of [3, 4]. To sample with N samples out of every M a multiband signal with bands as in (1) we must have

- $N f_0 \geq B_{eff}$
- $M f_0 \geq 2 f_{max}$
- $p_i = k_i f_0, \ i = 0, 1, \cdots 2L.$

The p_i are found by combining all of the numbers $\{-a_{L-1}, -a_{L-2}, \cdots, a_{-1}, a_0, a_1, \cdots, a_{L-1}\}$ and $\{-b_{L-1}, -b_{L-2}, \cdots, b_{-1}, b_0, b_1, \cdots, b_{L-1}\}$ such that $p_i = b_i - a_j$ or $p_i = b_i + b_j$ for some j. The most difficult of these conditions is the third. It says that to achieve the minimum rate we must pair the band edges in some fashion, and each pair must be an integer times f_0.

One approach is to pair corresponding a_i and b_i together. Thus if we could find f_0 such that each pair $b_i - a_i$ had f_0 as an integer factor we would have a solution. An important degree of freedom is available to us in the fact that there are many possible pairings. Exactly satisfying this condition is difficult in practice, since the a_i and b_i are real numbers, and pairing them such that any other real number is precisely an integer factor of all pairs is not possible in general. However, by examining a slightly larger set $Q \supset P$ we can find a solution easily. Thus we manipulate the positions of the bands slightly: given signals bandlimited to P we find a strategy to sample at minimum rate signals bandlimited to the larger set Q. The oversampling implied can be made arbitrarily small. Thus instead of achieving a minimum rate sampling solution for the given set of bands, we asymptotically approach that rate [4]

Theorem 2.1 *For signals bandlimited to P, where P is a finite union of open intervals and given any $\epsilon > 0$ there exists a set $Q \supset P$, such that the difference in the measures of the sets Q and P is less than ϵ and such that signals bandlimited to Q can be recovered from periodic non-uniform samples at the minimum rate.*

3. ASYMPTOTIC APPROACH TO MINIMUM RATE

The approach outlined in [4] shows how we can asymptotically approach the minimum sampling rate for multiband signals. Other solutions of the same problem are possible. The most common approach is based on a technique of slicing the spectrum, first used in a sampling context in [1].

The simplest embodiment of slicing [2, 1, 8] first divides the frequency interval between $-f_{max}$ and f_{max} into M equal intervals of size f_I. Clearly this gives $M f_I \geq 2 f_{max}$. Based on this slicing of the spectrum some of the M intervals will contain signal energy and some will not; denote by N the number of intervals that do contain signal energy. Then, using elementary multirate arguments, if we downsample the rate-$M f_I$ sampled sequence by M, no frequency will be hit by more than N copies of the spectrum. Thus, invoking Theorem 1.1, we can reconstruct the signal using a periodic non-uniform scheme where we retain only N samples out of every M. Clearly, within those intervals that contain signal energy there will generally be waste at the band edges. For a real signal with L bands, where we take slices of width f_I the wasted bandwidth will be

$$2 \sum_{i=0}^{L-1} a_i - \left\lfloor \frac{a_i}{f_I} \right\rfloor + \left\lceil \frac{b_i}{f_I} \right\rceil - b_i,$$

an upper bound on which is $4 L f_I$. Thus the only way to shrink the wasted bandwidth is to decrease f_I. If there are no constraints we can make approach the minimum rate asymptotically by making f_I arbitrarily small. In doing so however, M and hence N become arbitrarily large leading to a very complex reconstruction system. Variations on the slicing scheme, where for example we slice only the regions $(-f_{max}, -f_{min})$ and (f_{min}, f_{max}) can slightly increase the efficiency of the scheme [2].

3.1. Comparison with spectrum slicing

We demonstrate using the following example that slicing the spectrum [1, 2] can lead to very inefficient solutions. Consider a simple two band signal, *i.e.* $L = 2$, with $a_0 = \sqrt{2}/5$, $b_0 = \sqrt{3}/5$, $a_1 = 1 - \sqrt{2}/5$ and $b_1 = 2 - \sqrt{3}/5$. The effective bandwidth is 2 Hz. Observe that the minimum uniform rate to allow reconstruction would be $2 \cdot (2 - \sqrt{3}/5)$, giving that the wasted bandwidth is approximately 50 % of B_{eff}.

Using the approach of slicing the spectrum [1, 2, 8] the efficiency will be dominated the smallest of the signal bands. If we choose $f_I = 0.0123$ Hz we end up slicing the spectrum into $M = 270$ bands, only $N = 168$ of which contain signal energy. The wasted bandwidth is approximately 3.32 % of B_{eff}. This can be improved only at the cost of increased complexity. For example if we make $f_I = 0.00123$ Hz we
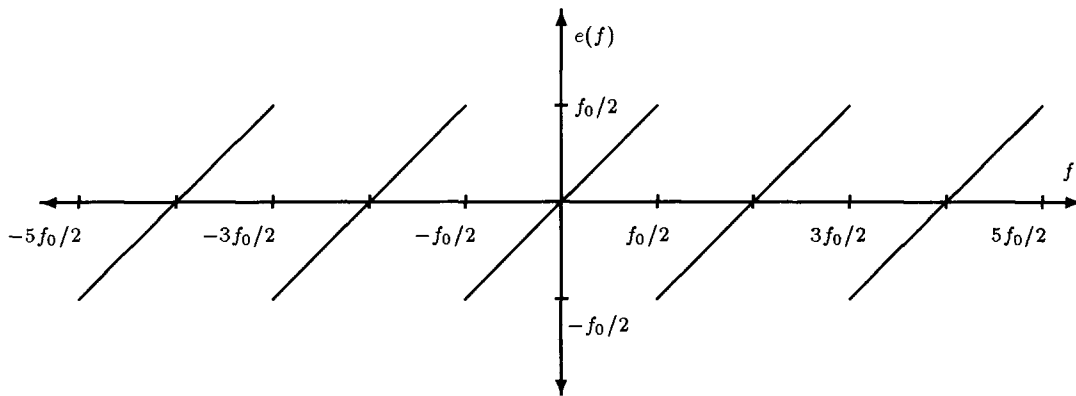
Figure 2: Periodic piecewise linear function $e(f)$.

end up slicing the spectrum into $M = 2690$ bands, only $N = 1630$ of which contain signal energy. The wasted bandwidth is approximately 0.245 % of B_{eff}. It can readily be seen that the slicing approach rapidly leads to complex reconstruction systems.

Alternatively, we can exploit the fact that to satisfy (4) we can pair band edges in any fashion we wish. For example, pairing b_0 and $-b_1$ and a_0 and $-a_1$, we can find a solution with $f_0 = 1$ Hz. We indeed then have $b_0 + b_1 = 2f_0$ and $a_0 + a_1 = f_0$. Thus (4) is solved with $N = 2$ and $M = 4$. The minimum rate is actually achieved in this case, so the efficiency is one. This serves to illustrate the important distinction between our approach and those derived from the slicing of bands used in [1, 2].

3.2. Complexity of reconstruction system

Periodic non-uniform reconstructions are in general not unique. In fact, even at the minimum rate, there are many sampling and reconstruction systems that lead to recovery. An important way of differentiating between various solutions is thus the complexity of the reconstruction system, which we now analyse.

The reconstruction system in general consists of a bank of N filters, the structure of which [3, 4] guarantees that only $M - N$ out of every M coefficients are non-zero. The filters are infinite in length, and some truncated or windowed version must be used. Thus the complexity of the entire reconstruction system is proportional to $N \cdot (M - N)$.

4. SPECIAL CASE RECONSTRUCTIONS

While the example discussed in Section 3.1 is obviously a special case it is worth emphasising that there is a considerable difference between the approach of [3, 4] and that of slicing the spectrum. This difference is exemplified by Theorem 1.2. This shows that we have considerable freedom in how to pair the band edges. If for any of these pairings (and there are $(2L)!$ pairings) we can have all pairs evenly divisible by f_0 (for any f_0) then we can achieve the minimum rate. This contrasts favorably with the slicing method which, while it approaches the minimum rate, does so at the cost of increasing complexity.

We will address in Section 5 strategies to pair the band edges to generate solutions that are efficient in the sense of giving low complexity solutions while keeping the wasted bandwidth low. We first examine two cases which are sufficiently important to merit separate discussion.

4.1. Minimum rate for equally sized bands

An important special case is that of a multiband signal that consists of equally sized bands. Such a case was considered in [7] and a means to allow sampling, under some restrictive assumptions was presented. We point out that a consequence of Theorem 1.2 is that whenever a multiband signal consists of equally sized bands we can always achieve the minimum rate. To do so we choose f_0 to be $b_i - a_i$ which is of course a constant. Thus condition (4) is satisfied with equality irrespective of the exact locations of the bands. In this case we will have $N = 2L$. We need in addition to know f_{max} in order to guarantee that $M f_0 \geq 2f_{max}$.

4.2. Spectrum blind reconstruction

A further implication of Theorem 1.2 is that in order to achieve or approach the minimum rate it suffices to know the sizes of the bands, and, again, knowing their locations is unecessary. Knowledge of f_{max} is however necessary. For example, if we know the band sizes $b_i - a_i$ for $i = 0, 1, \cdots L - 1$ then we can achieve the minimum rate if there is some f_0 that evenly divides all of these band sizes. A sampling strategy can be chosen and the reconstruction filters designed irrespective of the actual positions of the bands. For example suppose we have three bands of sizes 0.25 kHz, 1 kHz, and 2.75 kHz. It is clear that if we choose $f_0 = 0.25$ kHz we can satisfy the requirements of Theorem 1.2. We will then have $N = B_{eff}/f_0 = 32$. We cannot fix the sampling strategy without in addition knowing the maximum frequency in order to ensure that we pick M such that $M f_0 \geq 2f_{max}$. Observe that even if we allow the positions of the bands to change, the sampling strategy and the reconstruction procedure need not be redesigned provided that the band sizes and f_{max} remain unaltered. Obviously, if we cannot find an f_0 that evenly divides all of the bands, we cannot achieve the minimum rate, but can approach it, using the usual arguments.

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The first approach to spectrum blind reconstruction appears to be [2]. Their method is again based on slicing of the spectrum; it is less restrictive than the spectrum blind case that we have treated here, although suffers from the high complexity common to slicing approaches.

5. STRATEGIES TO CHOOSE THE PAIRINGS

The question of how to pair the band edges in general to solve (4), and how to choose f_0, N and M for best performance is an open one. Recall that to solve (4) we must pair each b_i with one of the a_j or one of the $-b_j$, and each pair must be an integer times f_0. Asymptotically, we can achieve any desired efficiency by making f_0 small. In practice, however, we probably wish to minimize the spectral waste, but we might also want to constrain N and M to be moderate also.

5.1. Redundancy constrained design

We consider the case where the allowable waste of bandwidth is fixed as ϵ. That is we can tolerate $Nf_0 < B_{eff} + \epsilon$. We wish to achieve this with the smallest possible N and M. Examining all possible band pairings becomes prohibitive for a large number of bands. We point out that in the argument of [4] to approach the minimum rate we incur a maximum bandwidth waste of f_0 for each band pair (thus we chose $f_0 < \epsilon/(2L)$ where ϵ was the total allowable waste). The larger the pairs in general, the larger the possible values of f_0 that we can try. If even one of the pairs is very small, we will probably incur a large waste unless we choose f_0 to be very small. We already discussed the merits of keeping f_0 as large as possible. Our approach then is to seek a pairing where all pairs are relatively large. We have found the following scheme to work well, but make no claims of optimality.

- Pair a_i with $-a_{L-i-1}$ and b_i with $-b_{L-i-1}$ for $i = 0, 1, \cdots, L/2 - 1$.
- Let $d_i = a_i + a_{L-i-1}$ and $d_{L/2+i} = b_i + b_{L-i-1}$ $i = 0, 1, \cdots, L/2 - 1$.
- Order and reindex the d_i so that: $d_0 \leq d_1 \leq \cdots \leq d_{L-1}$.
- Try $f_0 = d_{L-1}/p$ for $p = 1, 2, 3, \cdots$ until

$$\sum_i \lceil d_i/f_0 \rceil - d_i < \epsilon.$$

Compare with the case where we wish to use a slicing scheme, and limit the wasted bandwidth to ϵ. In this case we would have essentially only one degree of freedom. We choose some start value of f_I and keep decreasing f_I until $Nf_0 < B_{eff} + \epsilon$. In general this leads to more complex solutions.

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