OVERSAMPLED FILTER BANKS: OPTIMAL NOISE SHAPING, DESIGN FREEDOM, AND NOISE ANALYSIS*

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Abstract—We show that oversampled filter banks (FBs) offer more design freedom and less noise sensitivity than critically sampled FBs. We provide a parameterization of all synthesis FBs satisfying perfect reconstruction for a given oversampled analysis FB, and we derive bounds and expressions for the variance of the reconstruction error due to noisy subband signals. Finally, we introduce noise shaping in oversampled FBs and calculate the optimal noise shaping system.

1 INTRODUCTION AND OUTLINE

Recent interest in oversampled filter banks (FBs) [1]-[5] is due to their increased design freedom, reduced noise sensitivity, and noise reducing properties. This paper presents an analysis of these advantages of oversampled FBs.

Section 2 investigates the *design freedom* in oversampled FBs. We show that, for a given analysis FB, the synthesis FB providing perfect reconstruction (PR) is not unique, and we present a parameterization of all PR synthesis FBs [1, 2].

Section 3 presents a *noise analysis* for oversampled FBs. We derive bounds on the variance of the reconstruction error caused by noisy subband signals [1, 2], and we discuss the dependence of the error on the oversampling factor. A signal space interpretation of noise reduction is given, and the minimum norm synthesis FB is shown to minimize the error.

Finally, Section 4 proposes and analyzes the use of *noise shaping* in oversampled FBs. The optimal noise shaping system is derived, and a significant reduction of error variance is observed.

2 DESIGN FREEDOM

We consider a uniform FB [6, 7] with N channels (subbands), subsampling factor M in each channel, analysis filters $h_k[n] \leftrightarrow H_k(z)$, and synthesis filters $f_k[n] \leftrightarrow F_k(z)$ ($k = 0, 1, .., N-1$). The FB is said to be *critically sampled* or *maximally decimated* if $N = M$ and *oversampled* if $N > M$. The polyphase decompositions [6, 7] of the analysis and synthesis filters read $H_k(z) = \sum_{n=0}^{M-1} z^n E_{k,n}(z^M)$ and $F_k(z) = \sum_{n=0}^{M-1} z^{-n} R_{k,n}(z^M)$, respectively, with the polyphase components

$$E_{k,n}(z) = \sum_{m=-\infty}^{\infty} h_k[mM-n]\, z^{-m}, \qquad n = 0, 1, .., M-1$$

$$R_{k,n}(z) = \sum_{m=-\infty}^{\infty} f_k[mM+n]\, z^{-m}, \qquad n = 0, 1, .., M-1.$$

The $N \times M$ analysis polyphase matrix $\mathbf{E}(z)$ and the $M \times N$ synthesis polyphase matrix $\mathbf{R}(z)$ are defined as $[\mathbf{E}(z)]_{k,n} = E_{k,n}(z)$ and $[\mathbf{R}(z)]_{n,k} = R_{k,n}(z)$, respectively.

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A FB (critically sampled or oversampled) satisfies the perfect reconstruction (PR) property $\hat{x}[n] = x[n]$ if and only if [6, 7, 1, 2, 3]

$$\mathbf{R}(z)\,\mathbf{E}(z) = \mathbf{I}_M, \tag{1}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. In the critically sampled case ($N = M$), $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are square matrices and thus, assuming invertibility of $\mathbf{E}(z)$, (1) uniquely determines the synthesis polyphase matrix as $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$. In the oversampled case ($N > M$), the matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are rectangular and thus the solution $\mathbf{R}(z)$ of (1) is not unique. This freedom in designing the synthesis FB for given analysis FB is a desirable consequence of oversampling. Any solution of (1) is a left-inverse of $\mathbf{E}(z)$ that can be written as [8]

$$\mathbf{R}(z) = \hat{\mathbf{R}}(z) + \mathbf{U}(z)\left[\mathbf{I}_N - \mathbf{E}(z)\,\hat{\mathbf{R}}(z)\right]. \tag{2}$$

Here, $\hat{\mathbf{R}}(z)$ is the para-pseudo-inverse of $\mathbf{E}(z)$, which is a particular solution of (1) defined as[1]

$$\hat{\mathbf{R}}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z),$$

and $\mathbf{U}(z)$ is an arbitrary $M \times N$ matrix satisfying $|[\mathbf{U}(e^{j2\pi\theta})]_{n,k}| < \infty$. The para-pseudo-inverse $\hat{\mathbf{R}}(z)$ corresponds to minimum norm synthesis filters, i.e., $\sum_{k=0}^{N-1} \|f_k\|^2$ is minimal among all synthesis FBs providing PR [5].

Eq. (2) provides a parameterization of the class of all PR synthesis polyphase matrices $\mathbf{R}(z)$ in terms of the MN entries $[\mathbf{U}(z)]_{n,k}$ that can be chosen arbitrarily. This parameterization can also be formulated in the time domain as

$$f_k[n] = \hat{f}_k[n] + u_k[n] - \sum_{l=0}^{N-1}\sum_{m=-\infty}^{\infty} \left\langle \hat{f}_k, h_{l,m} \right\rangle u_{l,m}[n].$$

Here, the $\hat{f}_k[n]$ denote the minimum norm synthesis filters (corresponding to $\hat{\mathbf{R}}(z)$), $u_k[n]$ is the filter with polyphase components $[\mathbf{U}(z)]_{n,k}$, i.e., $U_k(z) = \sum_{n=0}^{M-1} z^{-n}[\mathbf{U}(z^M)]_{n,k}$, and finally $u_{k,m}[n] = u_k[n-mM]$ and $h_{k,m}[n] = h_k^*[mM-n]$. Equivalently, (2) can also be formulated in the frequency domain as

$$F_k(z) = \hat{F}_k(z) + U_k(z) - \frac{1}{M}\sum_{i=0}^{M-1} \hat{F}_k(zW_M^i) \sum_{l=0}^{N-1} H_l(zW_M^i)\, U_l(z),$$

where $W_M = e^{-j\frac{2\pi}{M}}$.

[1] Here, $\tilde{\mathbf{E}}(z) = \mathbf{E}^H(1/z^*)$ (with superscript ^{H} denoting conjugate transposition) stands for the paraconjugate of $\mathbf{E}(z)$ [6].

3 NOISE ANALYSIS

In this section, we shall investigate the sensitivity of oversampled FBs to (quantization) noise $n_k[m]$ added to the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$ $(k = 0, 1, ..., N-1)$. Let us collect the noise signals $n_k[m]$ in the N-dimensional vector noise process $\mathbf{n}[m]$ that is assumed to be wide-sense stationary (WSS) and zero-mean. The $N \times N$ power spectral matrix of $\mathbf{n}[m]$ is defined as $\mathbf{S}_n(z) = \sum_{l=-\infty}^{\infty} \mathbf{C}_n[l]\, z^{-l}$ with the autocorrelation matrix $\mathbf{C}_n[l] = \mathcal{E}\{\mathbf{n}[m]\, \mathbf{n}^H[m-l]\}$, where $\mathcal{E}$ denotes the expectation operator [6].

Variance of reconstruction error. It is convenient to redraw the FB in the "polyphase domain" as shown in Fig. 1 [6]. Here, $\mathbf{x}(z) = (X_0(z)\ X_1(z)\ ...\ X_{M-1}(z))^T$ and $\hat{\mathbf{x}}(z) = (\hat{X}_0(z)\ \hat{X}_1(z)\ ...\ \hat{X}_{M-1}(z))^T$ with $X_n(z) = \sum_{m=-\infty}^{\infty} x[mM+n]\, z^{-m}$ and $\hat{X}_n(z) = \sum_{m=-\infty}^{\infty} \hat{x}[mM+n]\, z^{-m}$, and the noise $\mathbf{n}[m]$ is represented by its z-transform $\mathbf{n}(z) = \sum_{m=-\infty}^{\infty} \mathbf{n}[m]\, z^{-m}$. Assuming a PR FB, we have (see Fig. 1) $\hat{\mathbf{x}}(z) = \mathbf{x}(z) + \mathbf{R}(z)\, \mathbf{n}(z)$, so that the reconstruction error $e[n] = \hat{x}[n] - x[n]$ is represented by

$$\mathbf{e}(z) = \hat{\mathbf{x}}(z) - \mathbf{x}(z) = \mathbf{R}(z)\, \mathbf{n}(z). \tag{3}$$

The reconstruction error $e[n]$ is again WSS and zero-mean, with $M \times M$ power spectral matrix [6]

$$\mathbf{S}_e(z) = \mathbf{R}(z)\, \mathbf{S}_n(z)\, \tilde{\mathbf{R}}(z) \tag{4}$$

and variance [9, 6]

$$\sigma_e^2 = \frac{1}{M} \int_0^1 \mathrm{Tr}\left\{\mathbf{S}_e(e^{j2\pi\theta})\right\} d\theta, \tag{5}$$

where Tr denotes the trace operator.

Henceforth we make the idealized assumption that the noise signals $n_k[m]$ are uncorrelated and white with identical variances $\sigma_n^2 = \mathcal{E}\{|n_k[m]|^2\}$. It follows that $\mathbf{C}_n[l] = \sigma_n^2\, \mathbf{I}_N\, \delta[l]$ and $\mathbf{S}_n(z) = \sigma_n^2\, \mathbf{I}_N$ [6]. With (4) and (5), the error variance becomes

$$\sigma_e^2 = \frac{\sigma_n^2}{M} \int_0^1 \mathrm{Tr}\left\{\mathbf{R}(e^{j2\pi\theta})\, \mathbf{R}^H(e^{j2\pi\theta})\right\} d\theta. \tag{6}$$

Frame-theoretic analysis of noise sensitivity. We now assume that the FB corresponds to a *frame expansion* [1, 2] in the sense that (i) the synthesis functions $f_{k,m}[n] = f_k[n - mM]$ constitute a *frame* for the space of square-summable signals, with frame bounds $A > 0$ and $B < \infty$ [10], and (ii) the analysis functions $h_{k,m}[n] = h_k^*[mM - n]$ are chosen as the dual frame [10]. This guarantees PR[2] and potentially good numerical properties (characterized by the frame bound ratio B/A). Furthermore, it can be shown [1, 2] that the total energy of the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$ is bounded as $\frac{1}{B}\|x\|^2 \le \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 \le \frac{1}{A}\|x\|^2$. For $A = B$ (i.e. a tight frame) we have $\sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 = \frac{1}{A}\|x\|^2$, that is, energy conservation up to a constant factor, which means that the FB is paraunitary [1, 2].

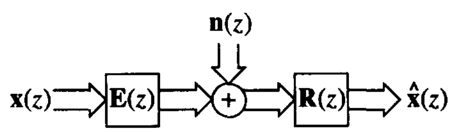

Figure 1. Adding noise to the subband signals.

[2]Choosing the analysis and synthesis functions to be dual frames corresponds to choosing $\mathbf{R}(z) = \hat{\mathbf{R}}(z)$ in (2) [1, 2].

The (tightest possible) frame bounds A and B of a FB providing a frame expansion are given by

$$A = \inf_{n=0,..,M-1,\ \theta\in[0,1)} \lambda_n(\theta), \qquad B = \sup_{n=0,..,M-1,\ \theta\in[0,1)} \lambda_n(\theta), \tag{7}$$

where $\lambda_n(\theta)$ denotes the eigenvalues of the matrix $\mathbf{R}(e^{j2\pi\theta})\, \mathbf{R}^H(e^{j2\pi\theta})$ [1, 2].

With $\mathrm{Tr}\left\{\mathbf{R}(e^{j2\pi\theta})\, \mathbf{R}^H(e^{j2\pi\theta})\right\} = \sum_{n=0}^{M-1} \lambda_n(\theta)$ and (7), it follows that $MA \le \mathrm{Tr}\left\{\mathbf{R}(e^{j2\pi\theta})\, \mathbf{R}^H(e^{j2\pi\theta})\right\} \le MB$. Inserting this in (6), we obtain

$$A \le \frac{\sigma_e^2}{\sigma_n^2} \le B\,, \tag{8}$$

i.e., the reconstruction error variance σ_e^2 is bounded in terms of the frame bounds A, B. Let us assume normalized analysis filters, i.e., $\|h_k\| = 1$ for $k = 0, 1, ..., N-1$. It can then be shown [1, 2] that $A \le \frac{1}{K} \le B$, where $K = \frac{N}{M}$ is the oversampling factor. Hence, for $A \approx B$ or equivalently $B/A \approx 1$, (8) implies that small perturbations in the subbands yield a small reconstruction error. The design of FBs with $B/A \approx 1$ (and additional desirable properties such as good frequency selectivity) is easier for larger oversampling factor.

For a paraunitary FB with $\|h_k\| = 1$ we have $A = B = \frac{1}{K}$, and hence (8) becomes

$$\frac{\sigma_e^2}{\sigma_n^2} = \frac{1}{K} \qquad \text{with } K = \frac{N}{M}\,. \tag{9}$$

Thus, in the paraunitary case the reconstruction error variance is inversely proportional to the oversampling factor K, which means that more oversampling entails more noise reduction. Such a "$1/K$ behavior" has previously been observed for oversampled A/D conversion [11], for tight frames in finite dimensional spaces [10, 12], and for reconstruction from a finite set of Weyl-Heisenberg (Gabor) or wavelet coefficients [10, 13]. Recently, under additional conditions, a $1/K^2$ behavior has been demonstrated for Weyl-Heisenberg frames [13, 14]. In Section 4, we shall propose noise shaping techniques which can do even better than $1/K^2$.

Noise reduction versus design freedom. Let us now consider an oversampled FB with $\mathbf{R}(z)$ chosen according to (2), i.e., $\mathbf{R}(z) = \hat{\mathbf{R}}(z) + \mathbf{U}(z)\left[\mathbf{I}_N - \mathbf{E}(z)\, \hat{\mathbf{R}}(z)\right]$, such that PR is guaranteed. Inserting (2) in (3), we obtain the following decomposition of the reconstruction error,

$$\mathbf{e}(z) = \mathbf{e}_{\mathcal{R}}(z) + \mathbf{e}_{\perp}(z)\,,$$

where

$$\mathbf{e}_{\mathcal{R}}(z) = \hat{\mathbf{R}}(z)\, \mathbf{n}(z)\,, \qquad \mathbf{e}_{\perp}(z) = \mathbf{U}(z)\, \mathbf{P}_{\perp}(z)\, \mathbf{n}(z)\,, \tag{10}$$

with $\mathbf{P}_{\perp}(z) = \mathbf{I}_N - \mathbf{E}(z)\, \hat{\mathbf{R}}(z)$. This can be interpreted as follows. Let $\mathcal{R} \subseteq [l^2(\mathbb{Z})]^N$ denote the range of the ***analysis FB operator*** that assigns to each input signal $x[n]$ the vector signal $\mathbf{v}[m]$ comprising the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$. That is, $\mathcal{R}$ is the linear space of all subband signal vectors $\mathbf{v}[m]$ obtained for square-summable input signals $x[n]$. Furthermore, let $\mathcal{R}^{\perp} \subseteq [l^2(\mathbb{Z})]^N$ be the orthogonal complement space [8] of $\mathcal{R}$. Then $\mathbf{P}_{\mathcal{R}}(z) = \mathbf{E}(z)\, \hat{\mathbf{R}}(z) = \mathbf{E}(z)\left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1} \tilde{\mathbf{E}}(z)$ and $\mathbf{P}_{\perp}(z) = \mathbf{I}_N - \mathbf{P}_{\mathcal{R}}(z)$ are the polyphase domain representations of the orthogonal projection operators on $\mathcal{R}$ and on $\mathcal{R}^{\perp}$, respectively.

The error component $\mathbf{e}_{\mathcal{R}}(z)$ in (10) can equivalently be written as $\mathbf{e}_{\mathcal{R}}(z) = \hat{\mathbf{R}}(z)\mathbf{P}_{\mathcal{R}}(z)\mathbf{n}(z)$, which shows that $\mathbf{e}_{\mathcal{R}}(z)$ is reconstructed from the subband noise component $\mathbf{P}_{\mathcal{R}}(z)\mathbf{n}(z)$ in $\mathcal{R}$. Similarly, $\mathbf{e}_{\perp}(z) = \mathbf{U}(z)\mathbf{P}_{\perp}(z)\mathbf{n}(z)$ is reconstructed from the subband noise component $\mathbf{P}_{\perp}(z)\mathbf{n}(z)$ in $\mathcal{R}^{\perp}$. Since the subband noise signals $n_k[m]$ were assumed uncorrelated and white, and since the spaces $\mathcal{R}$ and $\mathcal{R}^{\perp}$ are orthogonal, $\mathbf{e}_{\mathcal{R}}(z)$ and $\mathbf{e}_{\perp}(z)$ are uncorrelated. Hence, their variances, denoted respectively $\sigma^2_{\mathcal{R}}$ and $\sigma^2_{\perp}$, can simply be added to yield the overall reconstruction error variance [15],

$$\sigma_e^2 = \sigma^2_{\mathcal{R}} + \sigma^2_{\perp}.$$

The variance component $\sigma^2_{\mathcal{R}}$ is independent of the parameter matrix $\mathbf{U}(z)$, and thus of the particular $\mathbf{R}(z)$ chosen. The variance component $\sigma^2_{\perp}$, on the other hand, depends on $\mathbf{U}(z)$; it is an *additional* variance that will be zero if and only if $\mathbf{R}(z) = \hat{\mathbf{R}}(z)$. Indeed, it follows from (2) that $\mathbf{R}(z) = \hat{\mathbf{R}}(z)$ if and only if $\mathbf{U}(z)\mathbf{P}_{\perp}(z) \equiv \mathbf{0}$, in which case $\mathbf{e}_{\perp}(z) = \mathbf{U}(z)\mathbf{P}_{\perp}(z)\mathbf{n}(z) \equiv \mathbf{o}$ and thus also $\sigma^2_{\perp} = 0$. Hence $\hat{\mathbf{R}}(z)$, the para-pseudo-inverse of $\mathbf{E}(z)$ (corresponding to the minimum norm synthesis FB), yields the minimum reconstruction error variance $\sigma^2_{e,\min} = \sigma^2_{\mathcal{R}}$ among all PR synthesis polyphase matrices $\mathbf{R}(z)$. Using $\hat{\mathbf{R}}(z)$, all noise components orthogonal on the range space $\mathcal{R}$ are suppressed, while any other PR synthesis FB (which may have desirable properties such as improved frequency selectivity) leads to an additional error variance $\sigma^2_{\perp}$ since also noise components orthogonal on $\mathcal{R}$ are passed to the FB output. In this sense, there exists a tradeoff between design freedom and noise reduction.

Loosely speaking, the range space $\mathcal{R}$—and thus also the fixed noise component $\sigma^2_{\mathcal{R}}$—becomes "smaller" for increasing oversampling factor $K = N/M$. This explains why more oversampling tends to result in better noise reduction.

4 OPTIMAL NOISE SHAPING

The noise reduction in oversampled FBs can be further increased by means of noise shaping techniques that generalize noise shaping coders for oversampled A/D converters [16]. We here propose a noise shaping system cradled between the analysis FB ($\mathbf{E}(z)$) and the synthesis FB ($\hat{\mathbf{R}}(z)$; note that we use the minimum norm synthesis FB), and represented by the $N \times N$ transfer matrix $\mathbf{G}(z)$ (see Fig. 2). Modeling the quantizer in Fig. 2 by additive noise $\mathbf{n}(z)$ (cf. Fig. 1), it is readily shown that the reconstruction error is given by

$$\mathbf{e}(z) = \hat{\mathbf{R}}(z)\mathbf{G}(z)\mathbf{n}(z). \qquad (11)$$

Again assuming uncorrelated and white noise signals, i.e., $\mathbf{S}_n(z) = \sigma_n^2\mathbf{I}_N$, the reconstruction error variance is

$$\sigma_e^2 = \frac{\sigma_n^2}{M}\int_0^1 \mathrm{Tr}\left\{\hat{\mathbf{R}}(e^{j2\pi\theta})\mathbf{G}(e^{j2\pi\theta})\mathbf{G}^H(e^{j2\pi\theta})\hat{\mathbf{R}}^H(e^{j2\pi\theta})\right\}d\theta. \qquad (12)$$

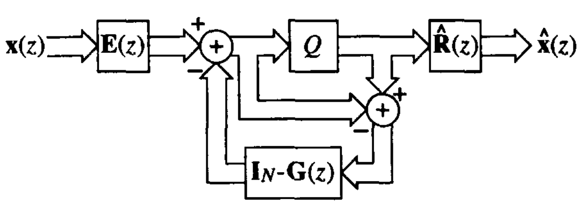

Figure 2. Oversampled FB with noise shaping. (The box labeled Q denotes the quantizer.)

Without further constraints, the noise could be completely removed using the orthogonal projection system $\mathbf{G}(z) = \mathbf{P}_{\perp}(z) = \mathbf{I}_N - \mathbf{E}(z)\hat{\mathbf{R}}(z)$. Indeed, inserting in (11) it follows with $\hat{\mathbf{R}}(z)\mathbf{E}(z) = \mathbf{I}_M$ that $\mathbf{e}(z) \equiv \mathbf{o}$. This noise shaper projects the noise onto $\mathcal{R}^{\perp}$, and the projected noise is then suppressed by the minimum norm synthesis FB $\hat{\mathbf{R}}(z)$.

Optimal noise shaper. Unfortunately, the above ideal noise shaper is inadmissible as it leads to a noncausal feedback loop system $\mathbf{I}_N - \mathbf{G}(z)$. Therefore, we hereafter constrain the noise shaping system to be a causal FIR system,

$$\mathbf{G}(z) = \mathbf{I}_N + \sum_{l=1}^{L}\mathbf{G}_l\, z^{-l},$$

resulting in a strictly causal feedback loop system $\mathbf{I}_N - \mathbf{G}(z)$. We now derive the optimal noise shaping system, i.e., the matrices $\mathbf{G}_l$ minimizing the reconstruction error variance σ_e^2 in (12). We assume a paraunitary FB with normalized, real-valued analysis filters ($h_k[n] \in \mathbb{R}$ and $\|h_k\| = 1$) of finite length $L_h = (P+1)M$ (with some $P \in \mathbb{N}$). We then have $\mathbf{E}(z) = \sum_{r=0}^{P}\mathbf{E}_r\, z^{-r}$ where $[\mathbf{E}_r]_{i,j} = h_i[rM - j] \in \mathbb{R}$. After some manipulations the error variance is obtained as [17]

$$\sigma_e^2 = \frac{\sigma_n^2}{MK^2}\left[MK + \mathrm{Tr}\left\{\sum_{l=1}^{L}\left(\mathbf{\Gamma}_l\mathbf{G}_l^T + \mathbf{\Gamma}_l^T\mathbf{G}_l\right)\right\} + \mathrm{Tr}\left\{\sum_{l=1}^{L}\sum_{m=1}^{L}\mathbf{\Gamma}_{m-l}\mathbf{G}_l\mathbf{G}_m^T\right\}\right],$$

where $\mathbf{\Gamma}_l = \sum_{r=0}^{P}\mathbf{E}_r\mathbf{E}_{r-l}^T$. From this expression, it can be seen that choosing the order of the noise shaping system as $L = P+1$ is sufficient. Setting (cf. [18], Section 5.3) $\frac{\partial\sigma_e^2}{\partial\mathbf{G}_i} = \mathbf{0}$ for $i = 1, ..., L$, we obtain the linear system of equations

$$\sum_{l=1}^{L}\mathbf{\Gamma}_{i-l}\,\mathbf{G}_l = -\mathbf{\Gamma}_i\,, \qquad (13)$$

which has block Toeplitz form and can thus be solved efficiently using the multichannel Levinson recursion [19]. Indeed, the noise shaping considered here can be shown [17] to be closely related to multichannel linear prediction [19].

A simple example. Let us consider a simple paraunitary two-channel FB (i.e., $N = 2$) with $M = 1$ and, hence, oversampling factor $K = 2$. The analysis filters are the Haar filters $H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$ and $H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$, and the minimum norm synthesis filters are $\hat{F}_0(z) = \frac{1}{2}\tilde{H}_0(z)$ and $\hat{F}_1(z) = \frac{1}{2}\tilde{H}_1(z)$. Without noise shaping, we obtain $\sigma_e^2 = \sigma_n^2/2$, which is consistent with the $1/K$ result (9).

With (13), the optimal noise shaping system of order $L = 1$ is obtained as $\mathbf{G}(z) = \mathbf{I}_2 + \mathbf{G}_1\, z^{-1}$ with

$$\mathbf{G}_1 = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix},$$

and the minimal error variance is obtained as $\sigma_e^2 = \sigma_n^2/4$. Thus, the variance has been reduced by a factor of 2. It is instructive to compare this result with the optimum noise shaping system $\mathbf{G}^D(z)$ of order $L = 1$ obtained under the constraint that $\mathbf{G}(z)$ is a diagonal matrix (i.e., the redundancy between the two channels is not exploited); here,

$$\mathbf{G}_1^D = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\sigma_e^2 = \frac{3}{8}\sigma_n^2$. Thus, as expected, failing to exploit the interchannel redundancy leads to a larger error variance.

The transfer functions $\hat{F}_0(z)$, $\hat{F}_1(z)$ of the synthesis FB and the transfer functions $G_{00}(z)$, $G_{11}(z)$ of the noise shaping filters in the diagonal of $\mathbf{G}(z)$ (the same as in the diagonal of $\mathbf{G}^D(z)$) are depicted in Fig. 3.

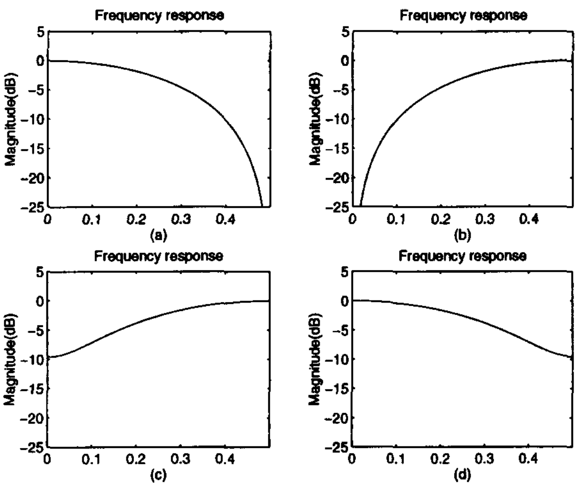

Figure 3. Synthesis filters and noise shaping filters in an oversampled two-channel FB: (a) $\hat{F}_0(z)$, (b) $\hat{F}_1(z)$, (c) $G_{00}(z)$, (d) $G_{11}(z)$.

It can be seen that the noise shaping system $G_{00}(z) = 1 - \frac{1}{2}z^{-1}$ (operating in the lowpass channel) attenuates the noise at low frequencies (note that $\hat{F}_0(z)$ attenuates high frequencies), whereas the noise shaping system $G_{11}(z) = 1 + \frac{1}{2}z^{-1}$ (operating in the highpass channel) attenuates the noise at high frequencies (note that $\hat{F}_1(z)$ attenuates low frequencies).

Simulation results. For three paraunitary odd-stacked cosine modulated FBs [17] with $N=16$, $L_h = 81$ (length of prototype), and $M=8$, 4, and 2, respectively (i.e., oversampling factors $K=2$, 4, and 8, respectively), Fig. 4 shows the normalized error variance $10\log(\frac{\sigma_e^2}{\sigma_n^2})$ as a function of the noise shaping system's order L. Note that for increasing L the error variance decreases up to a certain point, after which it remains constant.

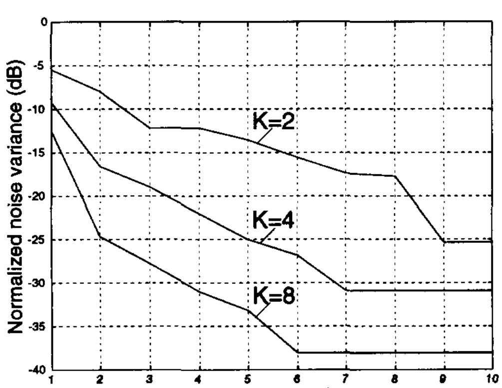

Fig. 4. Normalized error variance $10\log(\sigma_e^2/\sigma_n^2)$ as a function of the noise shaping system's order L.

5 CONCLUSION

We have shown that oversampled FBs feature increased design freedom and improved noise immunity. The latter property allows a coarser quantization of the subband signals. We introduced oversampled noise shaping subband coders that exploit intrachannel and interchannel redundancies to yield a substantial noise reduction. A rate-distortion analysis [20] of source coding using oversampled FBs is an interesting direction of further research; first results on this topic (without noise shaping) have been reported in [21].

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References

[1] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, "Frame-theoretic analysis and design of oversampled filter banks," in *Proc. IEEE ISCAS-96*, vol. 2, (Atlanta, GA), pp. 409–412, May 1996.

[2] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, "Frame-theoretic analysis of filter banks," *submitted to IEEE Trans. Signal Processing*, Feb. 1996.

[3] Z. Cvetković and M. Vetterli, "Oversampled filter banks," *IEEE Trans. Signal Processing*, to appear.

[4] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, "Oversampled FIR and IIR DFT filter banks and Weyl-Heisenberg frames," in *Proc. IEEE ICASSP-96*, vol. 3, (Atlanta, GA), pp. 1391–1394, May 1996.

[5] A. J. E. M. Janssen, "Density theorems for filter banks," Tech. Rep. 6858, Philips Research Laboratories Eindhoven, The Netherlands, April 1995.

[6] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs (NJ): Prentice Hall, 1993.

[7] M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*. Englewood Cliffs (NJ): Prentice Hall, 1995.

[8] T. Kailath, *Linear Systems*. Englewood Cliffs (NJ): Prentice Hall, 1980.

[9] R. A. Haddad and K. Park, "Modeling, analysis, and optimum design of quantized M-band filter banks," *IEEE Trans. Signal Processing*, vol. 43, pp. 2540–2549, Nov. 1995.

[10] I. Daubechies, *Ten Lectures on Wavelets*. SIAM, 1992.

[11] T. A. C. M. Claasen, W. F. G. Mecklenbräuker, and N. V. Hurck, "Signal processing method for improving the dynamic range of A/D and D/A converters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 28, no. 5, pp. 529–538, 1980.

[12] V. K. Goyal, "Quantized overcomplete expansions: Analysis, synthesis and algorithms," Master's thesis, University of California at Berkeley, July 1995.

[13] N. J. Munch, "Noise reduction in tight Weyl-Heisenberg frames," *IEEE Trans. Inf. Theory*, vol. 38, pp. 608–616, March 1992.

[14] Z. Cvetković and M. Vetterli, "Overcomplete expansions and robustness," in *Proc. IEEE TFTS-96*, (Paris, France), pp. 325–328, June 1996.

[15] L. L. Scharf, *Statistical Signal Processing*. Reading (MA): Addison Wesley, 1991.

[16] S. K. Tewksbury and R. W. Hallock, "Oversampled, linear predictive and noise-shaping coders of order N>1," *IEEE Trans. Circuits and Systems*, vol. 25, pp. 436–447, July 1978.

[17] H. Bölcskei, *Oversampled filter banks: Theory, design, and applications*. PhD thesis, Vienna University of Technology, in preparation.

[18] A. Weinmann, *Uncertain Models and Robust Control*. Vienna, Austria: Springer, 1991.

[19] S. M. Kay, *Modern Spectral Estimation*. Englewood Cliffs (NJ): Prentice Hall, 1988.

[20] T. Berger, *Rate Distortion Theory*. Englewood Cliffs (NJ): Prentice Hall, 1971.

[21] R. Heusdens, *Overlapped transform coding of images: Theory, application and realization*. PhD thesis, Delft University of Technology, Delft, The Netherlands, March 1997.