PARAMETERIZATION OF SYMMETRIC MULTIWAVELETS

Peter Rieder

Institute of Network Theory & Circuit Design
Technical University of Munich
Arcisstr. 21, 80333 Munich, Germany
e-mail: peri@nws.e-technik.tu-muenchen.de

ABSTRACT

In this paper multiwavelets based on two scaling functions are discussed. They exhibit the following properties: compact support, symmetry and orthogonality as well as a good frequency resolution. Lattice structures do not only offer the possibility to implement these multiwavelet transforms, the lattice rotation angles also can be used in order to parameterize all multiwavelets of a certain length. Here we search for optimal multiwavelets with respect to regularity, vanishing moments, frequency behavior (stopband attenuation) and also take a simple implementation into consideration.

1. INTRODUCTION

In recent years wavelet transforms have gained a lot of interest in many application fields, e.g. image and speech processing [2], denoising or solving differential and integral equations [1]. Different variations of wavelet bases have been presented, whereby most attention was focussed on single wavelet transforms [3]. Single wavelet transforms are based on one scaling function $\Phi(t)$ and one wavelet function $\Psi(t)$, which meet the following dilation equations:

$$\Phi(t) = \sum_{k=0}^{n-1} g_k \Phi(2t-k); \qquad \Psi(t) = \sum_{k=0}^{n-1} h_k \Phi(2t-k).$$

The discrete coefficients g_k and h_k define the discrete wavelet transform and the complementary wavelet filters $G(z) = \sum_k g_k z^{-k}$ (low pass) and $H(z) = \sum h_k z^{-k}$ (high pass), and also appear in the wavelet basis matrix $\boldsymbol{W} = \begin{bmatrix} g_k \\ h_k \end{bmatrix}$.

One possibility to implement orthogonal wavelet filters is using lattice structures. These lattice structures also allow a parameterization of orthogonal wavelet transforms. In Figure 1, a lattice structure is shown implementing wavelet filters of length $n = 4$. Depending on α, wavelets of $n = 4$ can be parameterized. In [3] versions showing a maximal amount of vanishing moments, or most regular wavelets were presented, wavelets with better frequency resolution were discussed in [5]. The property of a very simple implementation was embedded into the wavelet design in [7], whereby only very few CORDIC elementary steps were used requiring only a reduced number of shift and add operations for implementing the orthogonal filters. Allowing to preserve the signal's phase in the subband as well as an efficient processing at borders, symmetry is a desired property in image processing applications. However, it is impossible to design orthogonal and symmetric singlewavelets (except for the trivial Haar basis). Designing regular, orthogonal and symmetric wavelet systems is only possible by using several scaling functions and wavelets [10, 4, 6, 11, 8]. In [9] multiwavelets based on two scaling functions were presented being orthogonal *and* symmetric. They show a good frequency resolution (similar to M-band wavelets) and are quite regular. In this paper the approach of [9] is generalized, multiwavelets based on two scaling functions and wavelets are parameterized in the domain of the rotation angles of special lattice filters. Multiwavelet systems are designed with a maximal amount of vanishing moments, optimal regularity or stopband attenuation. Also a simple implementation is taken into consideration.

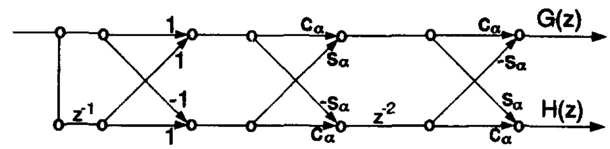


Figure 1: Lattice structure implementing a singlewavelet filter (n=4)

2. DESIGN OF SYMMETRIC MULTIWAVELETS

Multiwavelet systems using 2 scaling functions and 2 wavelets are based on 4 dilation equations, that are also represented by the basis matrix $\boldsymbol{W}$ of size $4 \times 4m$.

$$\Phi_v(t) = \sum_{l=1}^{2} \sum_{k=0}^{2m-1} g_{v,2k-1+l} \Phi_l(2t-k); \qquad v \in \{1,2\};$$

$$\Psi_v(t) = \sum_{l=1}^{2} \sum_{k=0}^{2m-1} h_{v,2k-1+l} \Phi_l(2t-k); \qquad v \in \{1,2\};$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}^U \\ \mathbf{W}^L \end{pmatrix} = \begin{pmatrix} g_{1,0} & g_{1,1} & \cdots & g_{1,4m-1} \\ g_{2,0} & g_{2,1} & \cdots & g_{2,4m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,4m-1} \\ h_{2,0} & h_{2,1} & \cdots & h_{2,4m-1} \end{pmatrix}$$

In [9] multiwavelets were designed by the algebraic design method. Thereby the properties of the continuous bases (i.e. orthogonality and vanishing moments) are converted to equations for the discrete coefficients of the basis matrix $\boldsymbol{W}$ resulting in a system of equations. In order to construct symmetric bases with good frequency resolution, a specially structured matrix $\boldsymbol{W}$ bases the approach also reducing the number of unknowns drastically. As example the structured basis matrix for $n = 2m = 6$ is shown:

$$\boldsymbol{W} = \begin{pmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 & a_2 & -b_2 & a_1 & -b_1 & a_0 & -b_0 \\ a_0 & -b_0 & -a_1 & b_1 & a_2 & -b_2 & -a_2 & -b_2 & a_1 & b_1 & -a_0 & -b_0 \\ b_0 & a_0 & b_1 & a_1 & b_2 & a_2 & -b_2 & a_2 & -b_1 & a_1 & -b_0 & a_0 \\ b_0 & -a_0 & -b_1 & a_1 & b_2 & -a_2 & b_2 & a_2 & -b_1 & -a_1 & b_0 & a_0 \end{pmatrix};$$

Solving the system of equations enables a wavelet system with $p = 3$ vanishing moments. The corresponding bases are called M_6^V in this paper, they are plotted in Figure 2. Their frequency characteristics are shown in Figure 3. Similar to the singlewavelet case, al-

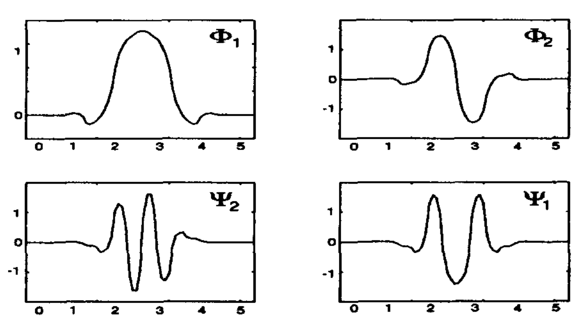


Figure 2: Multiwavelets of $p = 3$ and the corresponding scaling functions (version M_6^V)

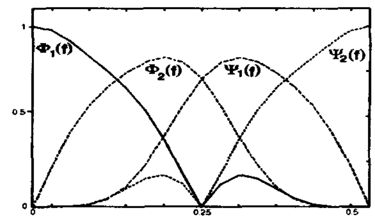


Figure 3: Frequency characteristics of the symmetric multiwavelets of p=3 and the corresponding scaling functions

ternatively to the solution with a maximal number of vanishing moments, other solutions (e.g. a more regular version) are of great interest. In order to find these bases a tool to parameterize all multiwavelets of certain compact support is required.

3. PARAMETERIZATION OF SYMMETRIC MULTIWAVELETS

Lattice structures are not only good tools for implementing multiwavelet transforms, also the parameterization of all multiwavelets of certain length is possible in the domain of the lattice rotation angles. Figure 4 shows a lattice structure implementing multiwavelet filters of length $n = 6$ ($G_v(z) = \sum_{i=0}^{n-1} g_{v,i} z^{-i}, H_v(z) = \sum_{i=0}^{n-1} h_{v,i} z^{-i}, v \in \{1,2\}$). Though arbitrary lengths are possible, throughout the paper we focus on the example of $n = 6$. Orthogonality is ensured by only using

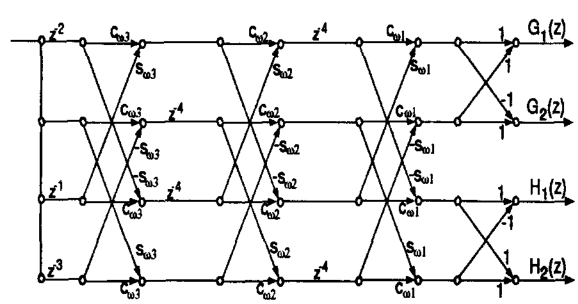


Figure 4: Lattice structure implementing multiwavelet filters of $n = 6$, ($c_{\omega i} = \cos\omega_i, s_{\omega i} = \sin\omega_i$)

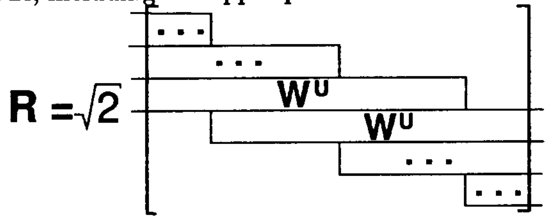
orthogonal rotations, a wavelet transform (at least one vanishing moment) is guaranteed by the constant sum of rotation angles

$$\omega_1 + \omega_2 + \omega_3 = 0 \qquad (1)$$

leading to a two–dimensional parameter space. In this two–dimensional parameter space, one can search for the optimal multiwavelet with respect to the desired properties.

Compact Support: If all parameters are zero, one gets the trivial Haar–based multiwavelets of support $n = 2$. Setting only $\omega_3 = 0^o$ results in bases with $n = 4$, whereby multiwavelets with approximation order $p = 2$ can be designed ($\omega_1 = -\omega_2 = 6.8221^o$). This version M_4^V is quite similar to the bases of Figure 2 and Figure 3.

Vanishing Moments and Regularity: A suitable tool for analysing multiwavelet systems is the matrix $\boldsymbol{R}$, including the upper part of the matrix $\boldsymbol{W}$.



If $\boldsymbol{R}$ has the eigenvalues 2^{-i}, $i = 0, \ldots, p-1$, the wavelet system has approximation order p. Within our two–dimensional parameter space the eigenvalue 1 is guaranteed. Since the multiwavelet system M_6^V of Figure 2 shows 3 vanishing moments (see also Figure 3, where $\frac{d^i\Psi_{1,2}}{df^i}|_{f=0} = 0, i = 0, 1, 2$) the respective matrix $\boldsymbol{R}$ has the eigenvalues 1, 1/2 and 1/4. The parameters of this solution are $\omega_1 = -4.3751^o, \omega_2 = 7.0801^o$ and $\omega_3 = -2.7050^o$. Evaluating the largest eigenvalue λ being different to 2^{-i} leads to an upper bound of the regularity of the wavelet system $r = -\log_2 \lambda$. For version M_6^V $r = 1.0637$. In order to improve the smoothness similar to the singlewavelet case giving up vanishing moments allows to maximize r and minimize λ, whereby always p vanishing moments must be guaranteed to achieve a regularity $r > p - 1$. The version M_6^R with optimal regularity requires rotation angles $\omega_1 = -19.4084^o, \omega_2 = 6.4524^o$ and $\omega_3 = 12.9559^o$. Thereby giving up one vanishing moment ($p = 2$) the smoothness can be improved to $r = 1.8221$. The continuous bases of solution M_6^R are plotted in Figure 5. The first numerical derivatives Φ_1' and Φ_2' of both scaling functions of Figure 6 show how the regularity is improved: The dotted lines representing M_6^V are rougher than the solid lines of version M_6^R.

Frequency Behaviour: In order to get the solution with the best stopband attenuation, the stobband norm of the scaling function $\Phi_1(f)$ at a certain scale is evaluated (because of the special structure of $\boldsymbol{W}$ improving $\Phi_1(f)$ is equivalent to improving the whole wavelet system). Choosing the parameters $\omega_1 = -29.9735^o, \omega_2 = 7.5935^o$ and $\omega_3 = 22.3800^o$ results in version M_6^F with optimal frequency behavior. The corresponding continuous bases of this version are plotted in Figure 7. Comparing the frequency behavior of version M_6^V (Figure 8, dotted line) with the stopband attenuation of

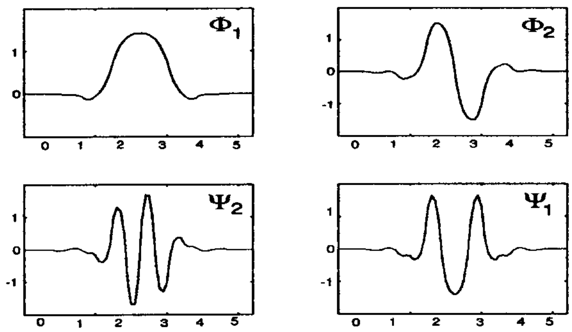


Figure 5: More regular multiwavelets of $p = 3$ and the corresponding scaling functions

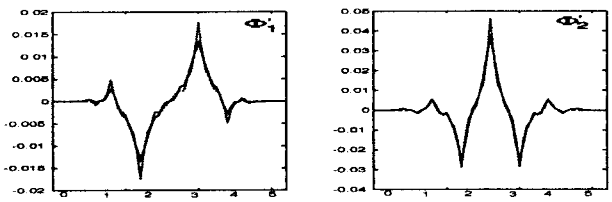


Figure 6: First numerical derivative of the scaling functions $\Phi_{1,2}$ of version M_6^V (dotted line) and version M_6^R (solid line)

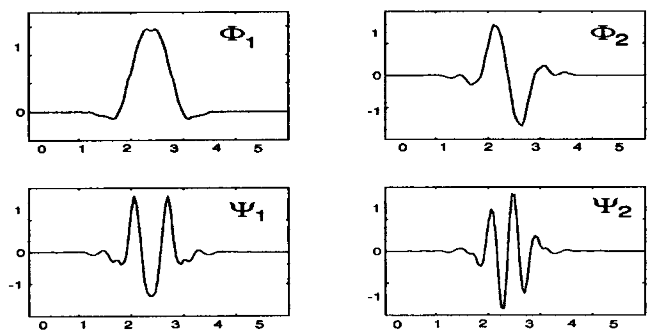


Figure 7: Multiwavelets ($n = 6$, $p = 1$) with improved frequency behavior and the corresponding scaling functions

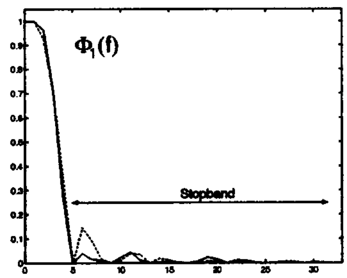


Figure 8: Frequency characteristics of Φ_1 of version M_6^V (dotted line) and version M_6^F (solid line)

version M_6^F (Figure 8, solid line) shows the achieved improvement.

Simple VLSI–Implementation: For the implementation of orthogonal singlewavelet transforms special 2×2–rotations can be implemented very efficiently using CORDIC–based elementary rotations $G(\alpha_k)$ [7]. They rotate by $\alpha_k = \arctan 2^{-k}$ and require only a few shift and add operations.

$$G(\alpha_k) = \frac{1}{\sqrt{1+2^{-2k}}} \begin{bmatrix} 1 & \sigma 2^{-k} \\ -\sigma 2^{-k} & 1 \end{bmatrix}$$

Thereby, double rotations $G(\pm\alpha_k)G(\mp\alpha_k)$ are required in order to simplify the scaling factor,

$$\frac{1}{\sqrt{1+2^{-2k}}^2} = (1-2^{-2k})(1+2^{-4k})(1+2^{-8k})\ldots$$

as well as in the parameterization scheme to conserve the constant sum of rotation angles. Also for the efficient implementation of the presented multiwavelet systems CORDIC–based approximate rotations can be used instead of the exact rotations without violating the orthogonality property and the first vanishing moment (1). Using one or a few of these simple CORDIC–based elementary rotations reduces the continuous parameter space to a reduced, discrete parameter space. In this reduced parameter space close approximations to the exact solutions are possible allowing a very simple implementation without loosing the performance of the transform. The lattice structure of the multiwavelet transform with minimized computational costs, approximating the exact transform M_4^V quite well is shown in Figure 9 ($\omega_1 = -\omega_2 = \frac{180}{\pi}\arctan 2^{-3} \approx 7.1^o$, $\omega_3 = 0^o$).

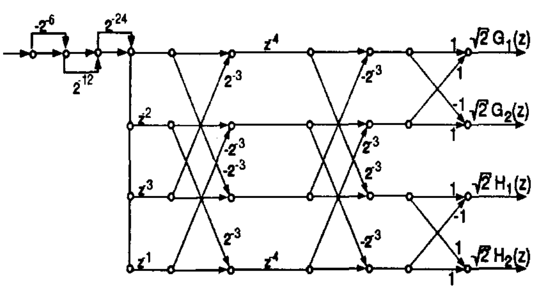


Figure 9: Lattice structure approximating the multiwavelet filters of $n = 4$ and $p = 2$ (version M_4^V)

4. CONCLUSION

In this paper symmetric and orthogonal multiwavelets are parameterized. The parameters are the rotation angles of the lattice structures implementing the stages of the wavelet transform. In the parameter space solutions with a maximal amount of vanishing moments were evaluated, as well as versions with improved regularity and frequency behavior. Also a simple implementation of the discrete transforms was taken into consideration. The design and parameterization of orthogonal, compactly supported wavelets is generalized to symmetric multiwavelets.

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