

BALANCED MULTIWAVELETS

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ABSTRACT

This paper deals with multiwavelets which are a recent generalization of wavelets in the context of multirate filter banks and their applications to signal processing and especially compression. By their inherent structure, multiwavelets are fit for processing multi-channel signals. First, we will recall some general results on multiwavelets and the convergence of the iterated matrix product. Then, we will define under what conditions we can apply systems based on multiwavelets to one-dimensional signals in a simple way. That means we will give some natural and simple conditions that should help in the design of new multiwavelets for signal processing. Finally, we will provide some tools in order to construct multiwavelets with the required properties, the so-called ‘balanced multiwavelets’.

1. INTRODUCTION

Wavelet constructions from iterated filter banks, as pioneered by Daubechies [3], have become a standard way to derive orthogonal and biorthogonal wavelet bases. The underlying filter banks are well studied, and thus, the design procedure is well understood. By the structure of the problem, certain issues are ruled out: e.g. impossibility of constructing orthogonal FIR linear phase filter banks. Nevertheless, by relaxing the requirement of time-invariance, it is easy to see that new solutions are possible. As mentioned in [12], such filter banks are closely related to some matrix 2-scale equations leading to multiwavelets.

2. MULTIWAVELETS

Similar to the wavelet case, the multiscaling function $\phi(t) := [\phi_0(t), \dots, \phi_{r-1}(t)]^T$ is solution of a 2-scale equation

$$\phi(t) = \sum_k \mathbf{M}[k] \phi(2t - k) \quad (1)$$

where now $\mathbf{M}[k]$ are $r \times r$ matrices of real coefficients. However, in the rest of the paper, for simplicity, we will concentrate on the case $r = 2$. The properties of the scaling function are strongly dependent on the spectral behavior of the refinement mask

$$\mathbf{M}(\omega) := \frac{1}{2} \sum_k \mathbf{M}[k] e^{-j\omega k} \quad (2)$$

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By defining the Fourier transform componentwise for vector-valued functions, the 2-scale equation converts to the equivalent form

$$\hat{\phi}(2\omega) = \mathbf{M}(\omega) \hat{\phi}(\omega) \quad (3)$$

and we can then derive the behavior of the scaling function by iterating this product [2].

In the wavelet case, $M(\omega)$ is a trigonometric polynomial satisfying the following two necessary constraints: (i) $M(0) = 1$ and (ii) $M(\pi) = 0$ for the iterated product to converge. The multiwavelet case is more tedious. As in [12], we define $D(\omega)$ the determinant of $\mathbf{M}(\omega)$, and $\{\lambda_0(\omega), \lambda_1(\omega)\}$ the eigenvalues of $\mathbf{M}(\omega)$. We also define

$$\mathbf{L} := \begin{pmatrix} \dots & \dots & M[1] & M[0] & \dots & \dots & \dots \\ \dots & M[3] & M[2] & M[1] & M[0] & \dots & \dots \\ \dots & M[5] & M[4] & M[3] & M[2] & \dots & \dots \end{pmatrix} \quad (4)$$

and

$$\Phi^{(n)}(\omega) := \mathbf{M}(\omega/2) \cdot \mathbf{M}(\omega/4) \dots \mathbf{M}(\omega/2^n) \cdot \Theta(\omega) \quad (5)$$

where $\Theta(\omega)$ is the normalized interpolation function

$$\Theta(\omega) := e^{-j\omega/2^{n+1}} \cdot \frac{\sin(\omega/2^{n+1})}{\omega/2^{n+1}} \quad (6)$$

Note that $\Phi^{(n)}(\omega)$ satisfies

$$\|\Phi_{i0}^{(n)}(\omega)\|_2^2 + \|\Phi_{i1}^{(n)}(\omega)\|_2^2 = 1 \quad (7)$$

given that we have orthogonality

$$\sum_k \mathbf{M}[k] \mathbf{M}[2l + k]^T = 2\delta_{0l} \mathbf{I} \quad \forall l \quad (8)$$

Also, $\Theta(\omega) \rightarrow 1$ for any finite ω and large n , and can thus be ignored. In the following, we will be interested in the limit

$$\Phi(\omega) := \lim_{n \rightarrow \infty} \Phi^{(n)}(\omega) = \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i) \quad (9)$$

Note that

$$\Phi(\omega) = \mathbf{M}(\omega/2) \cdot \Phi(\omega/2) \quad (10)$$

We then recall the results obtained in [6, 12, 2] about the convergence of the iterated matrix product:

Given an infinite matrix product of size 2 by 2

$$\Phi(\omega) := \prod_{i=1}^{\infty} M(\omega/2^i) \quad (11)$$

where $M(\omega)$ satisfies a matrix Smith-Barnwell condition

$$M(\omega)M^T(-\omega) + M(\omega + \pi)M^T(-\omega + \pi) = I. \quad (12)$$

a necessary condition for convergence to a scaling matrix $\Phi(\omega)$ such that $\Phi(0)$ is non-zero and bounded is

- (i) $M(0) = I$, $M(\pi) = 0$ (note that $\Phi(\omega)$ has rank 2)
- (ii) $M(0)$ has eigenvalue $\lambda_0(0) = 1$ and $|\lambda_1(0)| < 1$, $M(\pi)$ has rank 1 and satisfies $\mathbf{r}_0 \cdot M(\pi) = 0$ where $\mathbf{r}_0$ is a left eigenvector of $M(0)$ for the eigenvalue 1 (note that $\Phi(\omega)$ has then rank 1).

3. ROBUSTNESS OF EXPANSIONS

Recently, a surprising multiwavelet with symmetry, short compact support, orthogonality and also good approximation properties has been constructed by Geronimo, Hardin and Massopust (GHM) for the multiscaling functions [4] and by Strang and Strela for the multiwavelets [9]. Nevertheless, the theoretical results are somehow shadowed by computational drawbacks in applications like signal compression as mentioned in [11]. An important point to remember is that a multiwavelet filter bank (often abbreviated multifilter bank) is fundamentally a MIMO (multi-input multi-output) system that requires vectorization of the input signal which is usually one-dimensional to produce an input signal which is r -dimensional. However, due to some differences in the spectral behavior of the components of the scaling function vector, the 'lowpass' multifilter may have 'unbalanced' channels that complicate this vectorization. In that case, simple methods for the vectorization like splitting the input signal into blocks of size r lead to a mixing of coarse resolution and details creating strong oscillations in the reconstructed signal after compression as seen in Fig. 2. Namely, one of the important issues with wavelets in signal compression is the behavior of truncated series, i.e. robustness to truncation of the 'details' subbands. One would then expect some class of smooth signals to be well reproduced, i.e. one expect some kind of 'eigensignals' for the coarse approximation. For example, it would be reasonable to require $[1, 1, \dots, 1, \dots]^T$ to be preserved by the operator L i.e.

$$L[1, 1, \dots, 1, \dots]^T = [1, 1, \dots, 1, \dots]^T \quad (13)$$

However, most of the multiwavelets constructed so far don't even verify this simple requirement as illustrated in Fig. 1.

A solution proposed in [11] and generalized in [13] is to add some pre/post filtering of the input/output signal to adapt it to the spectral imbalance of the filter bank. A simple way of understanding prefiltering is to see it as a transform of the desired eigensignal $[1, 1, \dots, 1]$ into some genuine eigensignal associated to the eigenvalue 1 of L , for example, in the GHM case we have

$$L[1, \sqrt{2}, 1, \sqrt{2}, \dots, 1]^T = [1, \sqrt{2}, 1, \sqrt{2}, \dots, 1]^T \quad (14)$$

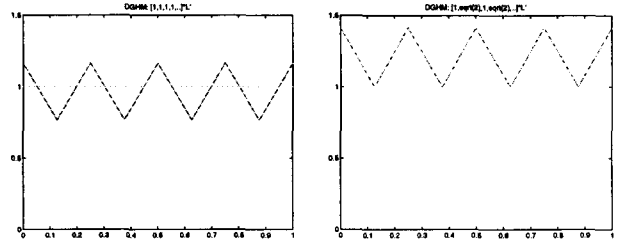


Figure 1: Left: Reproduction of the input signal $[1, 1, \dots, 1]$ by a GHM multiwavelet based filter bank without prefiltering, Right: reproduction of the eigensignal $[1, \sqrt{2}, 1, \sqrt{2}, \dots, 1]$

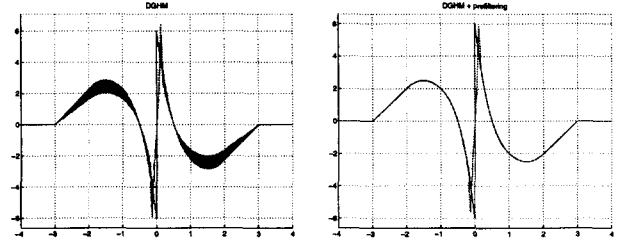


Figure 2: Truncation of the expansion associated to the details in a level 1 filter bank based on GHM multiwavelet. It shows on the left the poor robustness of systems based on GHM without pre/post filtering. The results are greatly enhanced with some pre/post filtering as seen on the graph on the right.

The results obtained (Fig. 2) using this 'trick' are of the same order as the ones obtained using a plain Daubechies filter bank with 4 taps. However, the new system constructed that way is no more orthogonal. Another way of doing pre/post filtering is to allow non critical sampling and to construct some projection of the input signal on V_0 . As mentioned in [13, 14], an issue is to then maintain orthogonality and critical sampling at the same time in the case of prefiltering. Thus, one may rather directly design orthogonal multiwavelets with good balance between the two scaling functions.

4. DESIGN OF BALANCED MULTIWAVELETS

In [12, 2], a necessary condition for the balancing of the scaling functions has been given: in the case $r = 2$, we need $[1, 1]^T$ to be a right eigenvector associated to the eigenvalue 1 of $M(0)$. This is easily understood by looking closely at (13). Furthermore, this implies that $\hat{\phi}(0) = [1, 1]^T$ i.e. ϕ_0, ϕ_1 are bona-fide lowpass scaling functions, and so the approximation rule on which the Mallat algorithm is based apply:

$$\int x(t)\phi_i(t-n)dt \approx x(n) \quad (15)$$

4.1. Complex Daubechies Multiwavelets

A simple way to construct balanced multiwavelets of arbitrary order is to derive them from the complex Daubechies

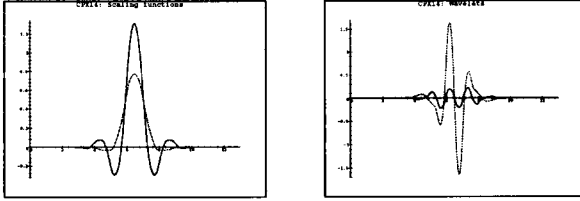


Figure 3: Highly regular Daubechies based Multiwavelets (same regularity as D14). Left: scaling functions, Right: multiwavelets.

filters. Daubechies filters are constructed using the half-band filter:

$$P(z) := c(1 + z^{-1})^N (1 + z)^N R(z) \quad (16)$$

such that $P(z) + P(-z) = 1$ with $R(e^{j\omega}) \geq 0$ and $R(e^{j\omega}) = R(e^{-j\omega})$. One get the usual Daubechies lowpass filters: $D_N(z) := (1 + z^{-1})^N B(z)$ where $B(z)$ is a spectral factor of $R(z)$ with real coefficients. We can't achieve orthogonality and symmetry with real coefficients, however by allowing complex coefficients in the spectral factorization, one can construct symmetric, orthogonal FIR filters [7]. Writing $[a[0], \dots, a[N], a[N], \dots, a[0]]$ for the lowpass filter, one can construct matrix coefficients:

$$A[i] := \begin{pmatrix} -Im(a[i]) & Re(a[i]) \\ Re(a[i]) & Im(a[i]) \end{pmatrix} \quad (17)$$

and the refinement mask is then with $z = e^{j\omega}$

$$M(\omega) := \frac{1}{2} \left(\sum_{i=0}^N A[i] z^{-i} + z^{-(N+1)} \sum_{i=0}^N A[N-i] z^{-i} \right) \quad (18)$$

The multifilter bank is clearly orthogonal and it is easily seen that the smoothness and approximation power of the Daubechies complex scaling functions and wavelets transfer to the multiscaling functions and multiwavelets. Namely, by defining

$$\psi(x) := \phi_1(x) + j\phi_0(x) \quad (19)$$

where $[\phi_0, \phi_1]$ is the multiscaling function associated to $M(\omega)$, we get that ψ verifies the 2-scale equation

$$\psi(x) = \sum_{k=0}^N a[k] \psi(2x-k) + \sum_{k=N+1}^{2N+1} a[2N+1-k] \psi(2x-k) \quad (20)$$

so ψ is the scaling function associated to the complex Daubechies filters, hence we get smoothness and approximation power for the multiscaling functions and the multiwavelets. We also derive that the multiscaling functions and multiwavelets are symmetric/antisymmetric as seen in Fig.3. However, this refinement mask when iterated doesn't converge properly because $M(0)$ has eigenvalues 1, -1 with eigenvectors $[1, 1]^T, [1, -1]^T$. We get only constrained convergence as defined in [5], hence the poor behavior of this multiwavelet in applications as seen in Fig 4.

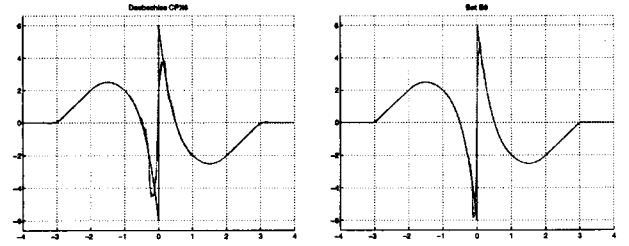


Figure 4: Robustness to truncation of the first order details subband of a 6x2 taps complex Daubechies based multiwavelet filter bank and the Chui based balanced multiwavelet with 8 2x2 taps filter bank.

4.2. Balancing of multiwavelets

Another interesting way of constructing balanced multiwavelets is to balance already existing multiwavelets like the ones constructed in [1] or [4]. The point is that we want $[1, 1]^T$ to be a right eigenvector associated with eigenvalue 1 of $M(0)$. The way to achieve this is to use the unitary matrix R such that

$$R^T M(0) R \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (21)$$

Defining the new refinement mask

$$P(\omega) := R^T M(\omega) R \quad (22)$$

and the new 2-scale equation

$$\hat{\psi}(2\omega) = P(\omega) \hat{\psi}(\omega) \quad (23)$$

We then verify that

$$\hat{\psi}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (24)$$

We notice that in the iteration, R^T and R cancel, except for the first and last term. The convergence of the matrix product for M imply the convergence for P and the smoothness and approximation power are therefore unchanged. However the symmetry of the scaling functions is usually lost. Nevertheless, the symmetry/antisymmetry of the multiwavelets can be maintained, by taking for the highpass refinement mask

$$Q(\omega) := N(\omega) R \quad (25)$$

where $N(\omega)$ is the highpass refinement mask associated to $M(\omega)$. Using Chui multiwavelets [1], we obtained orthogonal, compactly supported multiscaling functions / multiwavelets with symmetry and good approximation for the multiwavelets and also verifying the $[1, 1]^T$ right eigenvector condition (Fig. 5). This balanced multiwavelets have shown very good robustness in compression algorithm without any pre/post filtering (Fig. 4).

4.3. General design

A more general issue is then to describe some general design method for constructing bona-fide multiwavelets with

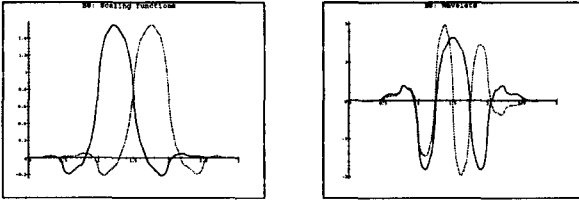


Figure 5: Balanced multiwavelet with 2^{nd} order of approximation. Left: scaling functions, Right: multiwavelets.

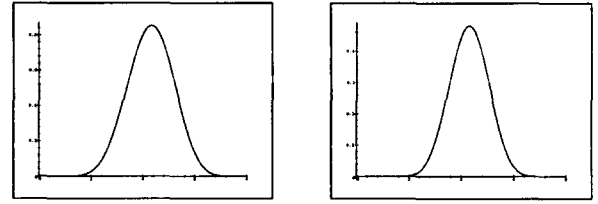


Figure 6: Highly regular scaling functions obtained from the Bat multiwavelet (support is $[0, 8]$).

all the desired properties. Recently Plonka and Strela proposed in [8, 10] a method to increase the approximation order of a given scaling function by what they called the 2-scale similarity transform. This transform applied to the refinement mask $M(\omega)$ determines a new scaling function with higher approximation order. This last one is derived from the new refinement mask $M_T(\omega)$ given by

$$M_T(\omega) := T(2\omega)M(\omega)T^{-1}(\omega) \quad (26)$$

where $T(\omega)$ is the transform matrix. Although this method showed some good results, as mentioned in [10], it is not clear how to maintain orthogonality and compact support at the same time. Moreover, this transform is not preserving the eigenvectors or even the eigenvalues of $M(0)$. So, we made some modification of this method by defining the new refinement mask $P(\omega)$ as

$$P(\omega) := T(\omega)M(\omega)T(\omega) \quad (27)$$

where now the transform matrices

$$T(\omega) := T_1^{-1}(\omega)T_2(2\omega) \quad (28)$$

verifies some weaker conditions than the ones required in [8]. This enables greater freedom in the design of the new refinement mask and allows especially to maintain the $[1, 1]^T$ eigenvector associated to the 1 eigenvalue condition on $P(0)$. As seen in Fig. 6, we constructed this way some highly regular biorthogonal balanced multiwavelets with compact support and symmetry starting from Chui's 1^{st} order balanced multiwavelet and using for example

$$T_1(\omega) = T_2(\omega) = \begin{pmatrix} (1-z)^2 & -z(1-z) \\ (1-z)^2 & (1-z)^2 \end{pmatrix} \quad (29)$$

where $z = e^{-j\omega}$. Nevertheless, the issue of maintaining the orthogonality during this process remains open.

5. CONCLUSION

After recalling some basic facts about multiwavelets, we described some tools to solve the problems we face applying multiwavelets in signal processing. However, some questions remain open. We still have to develop some systematic way to construct orthogonal balanced multiwavelets with the desired regularity. We also only dealt with the problem of first order balancing. The obvious generalization is the preservation of higher order polynomial signals by our multiwavelet based system. Some further developments and applications in the fields of one dimensional signal processing but also of array processing are then expected.

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