NON-WIENER EFFECTS IN LMS-IMPLEMENTED ADAPTIVE EQUALIZERS

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ABSTRACT

An adaptive transversal equalizer based upon the least-mean-square (LMS) algorithm, operating in an environment with a temporally correlated interference, can exhibit better steady-state mean-square-error (MSE) performance than the corresponding Wiener filter. This phenomenon is a result of the non-linear nature of the LMS algorithm and is obscured by traditional analysis approaches that utilize the independence assumption. We use a transfer function approach to quantify the MSE performance of the LMS algorithm and demonstrate that the degree to which LMS may outperform the corresponding Wiener filter is dependent on system parameters such as signal-to-noise ratio and the step-size parameter.

1. INTRODUCTION

Adaptive transversal equalizers are important components of digital receivers and primarily are used to mitigate the effects of intersymbol interference caused by multipath propagation and band limiting in the communication system [1]. The use of an adaptive equalizer as a method of interference suppression also is important, particularly in mobile digital radio systems.

The computationally efficient least-mean-square (LMS) adaptive algorithm [2] often is used in the implementation of the equalizer. Due to the non-linearity of the LMS algorithm, the optimum performance of the equalizer often is accessed using the Wiener realization of the adaptive filter [3]. The efficacy of this approach is based upon the argument that the LMS algorithm will result in greater mean-square-error (MSE) than the corresponding Wiener filter due to gradient noise on the adaptive filter weights. This argument is supported by traditional analysis approaches that invoke the independence assumption in which it is assumed that the current filter weight vector is statistically independent of the current tap data vector [4]. Then the resulting analytical expression of the MSE of the LMS algorithm is greater than the MSE produced by the Wiener filter. The expressions derived using the independence assumption generally have agreed closely with experimental results for a variety of adaptive filter applications such as the adaptive line enhancer and the adaptive noise canceler [5], when the LMS step-size parameter has a 'small value'.

However, recently it has been reported that an LMS-implemented adaptive equalizer operating with a temporally correlated interferer, can produce better probability-of-error performance than the corresponding Wiener filter [6]. Subsequent simulations have revealed the unexpected result that with the proper choice of the step-size parameter, the non-linear nature of the LMS algorithm can be exploited to generate MSE which is less than the Wiener MSE. As a result, an analysis of this problem cannot invoke the independence assumption.

To analyze this behavior, we utilize the transfer function approach first presented by Glover [7] for adaptive noise canceling of sinusoidal interferences and later generalized by Clarkson and White [8] to include deterministic interferences of arbitrary periodic nature and interferences which are stochastic. We present an analysis approach that generates an approximate expression of the steady-state MSE for the LMS algorithm. We specifically analyze equalizer performance for interference that is sinusoidal.

2. EQUALIZER PROBLEM

Fig. 1 represents the baseband adaptive equalizer structure to be analyzed. We present only the analysis of a symmetric, two-sided equalizer, even though the non-linear effects occur in one-sided equalizers as well. Vector quantities, such as the reference data vector $\mathbf{u}(k)$, are represented as

$$\mathbf{u}^T(k) = (u(k+N)\ \dots\ u(k)\ \dots\ u(k-N))^T, \quad (1)$$

where N is the number of leading and lagging taps, k is the time index, and T denotes transpose. The total number of taps is given by $L = 2N+1$. $\mathbf{u}(k)$ is decomposed into a sum of three statistically independent components as

$$\mathbf{u}(k) = \mathbf{s}(k) + \mathbf{x}(k) + \mathbf{n}(k), \quad (2)$$

where $\mathbf{s}(k)$ is the communication signal, $\mathbf{x}(k)$ is the interference vector, and $\mathbf{n}(k)$ is the noise vector. The communication signal and the noise are modeled as white processes with zero mean. The output of the adaptive filter $y(k)$ is given by the inner product of the filter weights and the data vector as

$$y(k) = \mathbf{w}^H(k)\mathbf{u}(k), \quad (3)$$

where H denotes Hermitian transpose. $y(k)$ is sent through a decision device to estimate $s(k)$, the symbol currently at the center tap.

During the convergence phase of the adaptive algorithm, the equalizer is in the training mode, in which the desired or primary input $d(k)$ is the error-free training sequence $s(k)$. During the communication phase, the equalizer is in the decision-directed mode in which the output of the decision device $\widehat{s}(k)$ is used as $d(k)$.

The weights of the equalizer are adapted by the complex LMS algorithm which uses the error sequence $e(k)$ to adjust the weights as [2]

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \mathbf{u}(k) e^*(k), \tag{4}$$

where μ is the step-size parameter, $e(k) = d(k) - y(k)$, and $*$ denotes complex conjugation.

Defining the $L \times L$ correlation matrix and the $L \times 1$ cross-correlation vector as

$$\mathbf{R} = E\left[\mathbf{u}(k)\mathbf{u}^H(k)\right] \quad \text{and} \quad \mathbf{p} = E\left[\mathbf{u}(k) d^*(k)\right], \tag{5}$$

the Wiener filter weights and associated MSE J_w are given by [4]

$$\mathbf{w}_w = \mathbf{R}^{-1}\mathbf{p} \quad \text{and} \quad J_w = \sigma_s^2 - \mathbf{p}^H \mathbf{w}_w, \tag{6}$$

where σ_s^2 is the communication signal power.

Under Gaussian assumptions, the Wiener filter provides the optimum MSE estimate $C_w(\mathbf{u}(k))$ of the parameter $s(k)$. However, this fact does not preclude the LMS algorithm from generating an estimate with less MSE. Due to the recursive and non-linear nature of the LMS algorithm, the estimate of $s(k)$ is a function of much more information than that utilized by the Wiener filter. The LMS algorithm not only utilizes more samples of the reference data, but, more importantly, it explicitly uses previously-detected symbols of the communication signal. The LMS estimator can be written abstractly as

$$y(k) = C_{\text{lms}}\left(\mathbf{u}(k)\ \mathbf{u}(k-1)\ \ldots\ ; d(k-1)\ d(k-2)\ \ldots\right). \tag{7}$$

In fact, this information is used in an equalizer structure consisting of a two-sided feed-forward transversal filter with an additional decision-feedback filter [3].

3. TRANSFER FUNCTION APPROXIMATION

We begin by decomposing the LMS filter weights into a sum of a steady-state, time-invariant mean component and a time-varying misadjustment component as

$$\mathbf{w}(k) = \mathbf{w}_s + \mathbf{w}_{\text{mis}}(k). \tag{8}$$

We begin steady-state filtering at time index $k = 0$. We also assume that $\mathbf{w}_{\text{mis}}(0) = \mathbf{0}$. This is equivalent to initializing the LMS algorithm with the mean filter. Using (4), we get the recursive equation for the misadjustment filter

$$\begin{aligned} \mathbf{w}_{\text{mis}}(k+1) &= \mathbf{w}_{\text{mis}}(k) + \mu \mathbf{u}(k) e^*(k) \qquad (9) \\ &= \mu \sum_{j=0}^{k-1} e^*(j) \mathbf{u}(j). \end{aligned}$$

Then the output process $y(k)$ can be written as

$$y(k) = \mathbf{w}_s^H \mathbf{u}(k) + \mu \sum_{j=0}^{k-1} e(j) \mathbf{u}^H(j) \mathbf{u}(k), \tag{10}$$

and the error process $e(k)$ as

$$e(k) + \mu \sum_{j=0}^{k-1} e(j) \mathbf{u}^H(j) \mathbf{u}(k) = d(k) - \mathbf{w}_s^H \mathbf{u}(k). \tag{11}$$

Equation (11) is a k^{th}-order recursive difference equation. Because the coefficients are stochastic, this equation is difficult to solve analytically. However, for wide sense stationary processes whose second-order moments can be estimated with time averages, Clarkson and White [8] propose using the approximation

$$\mathbf{u}^H(j)\mathbf{u}(k) \approx L r_u(k-j), \tag{12}$$

where $r_u(m)$ is the autocorrelation function of the reference process $u(k)$. Equation (11) then is approximated by a standard difference equation with constant coefficients as

$$e(k) + \mu L \sum_{j=0}^{k-1} r_u(k-j) e(j) = \widehat{d}(k), \tag{13}$$

where

$$\widehat{d}(k) = d(k) - \mathbf{w}_s^H \mathbf{u}(k). \tag{14}$$

We can then interpret the LMS error $e(k)$ as the output of a linear system with transfer function $H_E(z)$ given by [8]

$$H_E(z) = \frac{1}{1 + \mu L R(z)} \quad \text{with} \quad R(z) = \sum_{m=1}^{\infty} r_u(m) z^{-m}, \tag{15}$$

driven by $\widehat{d}(k)$. Because $\widehat{d}(k)$ also is a wide sense stationary process, the discrete power spectrum of the error process $e(k)$ is

$$S_E(z) = H_E(z) H_E^*(1/z^*) S_{\widehat{D}}(z). \tag{16}$$

The discrete power spectrum of the driving process is derived using (2) and (14) and is given by

$$\begin{aligned} S_{\widehat{D}}(z) = &\ (1 - W_s^*(z^*))(1 - W_s(1/z)) S_s(z) + \qquad (17) \\ &\ W_s^*(z^*) W_s(1/z)(S_x(z) + S_n(z)), \end{aligned}$$

where $S_s(z)$, $S_x(z)$, and $S_n(z)$ are the discrete power spectra of the communication signal, the interference process, and the noise process respectively, and $W_s(z)$ is the z-transform of the steady-state filter weights $\mathbf{w}_s$. We assume that, even if the equalizer is in the decision-directed mode, there are no decision errors. Using the assumption that both the communication signal and the noise process are white, we can replace $S_s(z)$ and $S_n(z)$ with the respective signal and noise powers σ_s^2 and σ_n^2 in (17). The approximation of the steady-state MSE of the LMS algorithm is the power of the process $e(k)$ given by

$$J_{\text{lms}} = \frac{1}{2\pi i} \oint_{|z|=1} S_E(z) z^{-1} dz. \tag{18}$$

4. CW INTERFERENCE

We apply this approach to derive the estimate of the LMS MSE for a complex sinusoidal interference scenario. To begin, we must know the steady-state mean weights in (8). Because $\mathbf{w}_s$ cannot be determined with this transfer function approach, it is necessary to use the Wiener weights $\mathbf{w}_w$. Although applying $W_s(z)$ to (18) would result in a more accurate estimate of MSE, the use of the Wiener transfer function has provided adequate results.

The interference vector is given by

$$\mathbf{x}^T(k) = \sigma_x e^{i(\omega_\Delta k + \phi)} \left(e^{i\omega_\Delta N}\ \ldots\ 1\ \ldots\ e^{-i\omega_\Delta N}\right)^T \tag{19}$$

where ω_Δ is the offset frequency of the interference, and ϕ is a random phase uniformly distributed between $-\pi$ and π. The autocorrelation function of $u(k)$ is given by

$$r_u(m) = \sigma_s^2\delta_m + \sigma_x^2 e^{i\omega_\Delta m} + \sigma_n^2\delta_m, \quad (20)$$

where δ_m is the Kronecker delta. Applying (20) to (15), and using the change of variable $z = \exp(i\omega)$ to (16) - (18), we get

$$|H_E(e^{i\omega})|^2 = \frac{2\,(1-\cos(\omega-\omega_\Delta))}{1+(\mu L\sigma_x^2-1)^2+2(\mu L\sigma_x^2-1)\cos(\omega-\omega_\Delta)}. \quad (21)$$

It is straightforward to find the frequency response $W_w(e^{i\omega})$ of the Wiener filter for this case. Then, (18) can be solved analytically to get [9]

$$J_{\text{lms}} = \frac{2}{1-A^2}\left(B - CA - DA^L\right), \quad (22)$$

where $A = 1-\mu L\sigma_x^2$, $B = C+D$, $C = \sigma_s^2\sigma_n^2/\left(\sigma_s^2+\sigma_n^2\right)$, $D = \left(\sigma_s^2\sigma_x^2/\left(\sigma_s^2+\sigma_n^2+L\sigma_x^2\right)\right)^2/\left(\sigma_s^2+\sigma_n^2\right)$, and $-1 < A < 1$. This condition on the variable A is required to make the filter $H_E(z)$ stable. However, it may not be adequate to ensure stability of the LMS algorithm. Using this equation, the MSE-optimum step size parameter can be found as

$$\mu_{\text{opt}} = \frac{1}{L\sigma_x^2}\left(1-\frac{1}{C}\left(B-\sqrt{B^2-C^2}\right)\right). \quad (23)$$

It also is informative to examine the theoretical power spectral density (PSD) of the LMS output process $y(k)$ which is given by

$$S_Y(e^{i\omega}) = |1+\left(W_w^*(e^{-i\omega})-1\right)H_E(e^{i\omega})|^2\sigma_s^2 + \quad (24)$$
$$|W_w^*(e^{-i\omega})H_E(e^{i\omega})|^2\sigma_n^2.$$

Numerical Examples: To demonstrate the validity of these results, theoretical LMS performance is compared to estimated performance derived experimentally via Monte Carlo simulations. The communication signal $s(k)$ is simulated as a quadrature phase shift keyed (QPSK) signal in which the mutually independent in-phase and quadrature components take values $+1$ and -1 with equal probability ($\sigma_s^2 = 2$).

Fig. 2 is a plot of MSE as a function of the step-size parameter μ for equalizer tap length $L = 51$, signal-to-noise ratio 25dB (SNR $= \sigma_s^2/\sigma_n^2$), and signal-to-interference ratio -20dB (SIR $= \sigma_s^2/\sigma_x^2$). The estimated LMS MSE obtained during the training phase and the MSE obtained during the decision-directed mode are plotted. Clearly, the MSE performance improvement of the LMS algorithm over the Wiener filter can be significant with the proper choice of μ. Also, close agreement with theory is observed. The optimum choice of μ determined from (23) is $\mu_{\text{opt}} = 3.78\times10^{-5}$, which agrees with the figure. Fig. 2 also is interesting because it contradicts conventional wisdom in adaptive filter theory in which a smaller step-size parameter μ is associated with less MSE. This is not the case for $\mu < \mu_{\text{opt}}$. However, as expected, the LMS MSE performance approaches that of the Wiener filter as $\mu \to 0$.

MSE as a function of SNR is shown in Fig. 3. The number of taps is $L = 51$ and SIR = -20dB. The step-size parameter is μ_{opt}. Also, the MSE of a Wiener filter with a decision feedback filter is included [3]. This figure demonstrates that for low to moderate SNR, the LMS-implemented equalizer achieves the MSE-performance of a Wiener filter with decision-feedback without explicitly incorporating a feedback structure.

Fig. 4 is a sequence of estimated and theoretical PSD's of the equalizer output process $y(k)$ operating in the decision-directed mode, with interferer offset frequency $\omega_\Delta = 1.5$ and parameters as in Fig. 2 for three values of μ. Fig. 4a) represents 'small' μ in which LMS performance is similar to that of the Wiener filter. The notching effect nulls the interference at the expense of inducing intersymbol interference in the communication signal $s(k)$. However, Fig. 4b) is the result of using μ_{opt}. The PSD in this case is seen to be almost flat, suggesting that interference nulling is occurring without the expense of significant distortion of $s(k)$. Fig. 4c) is for 'large' μ where spectral distortion is seen to increase again.

5. CONCLUSION

We have demonstrated the unexpected fact that an adaptive equalizer implemented with the LMS algorithm can have better interference nulling capabilities and exhibit better steady-state MSE performance than the corresponding Wiener filter. This non-Wiener effect is important because it conflicts with conventional wisdom in which it is assumed that LMS MSE exceeds that of the 'optimal' Wiener filter. The LMS algorithm achieves this improvement in performance by effectively incorporating information not used by the Wiener filter.

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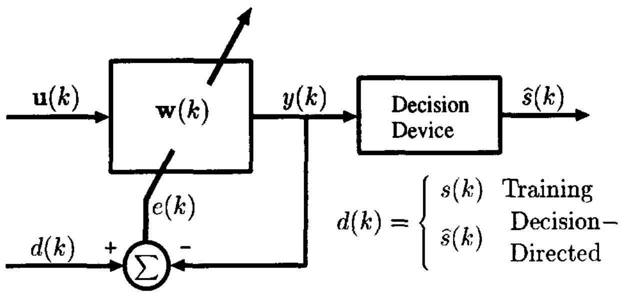

Figure 1. Adaptive equalizer structure.

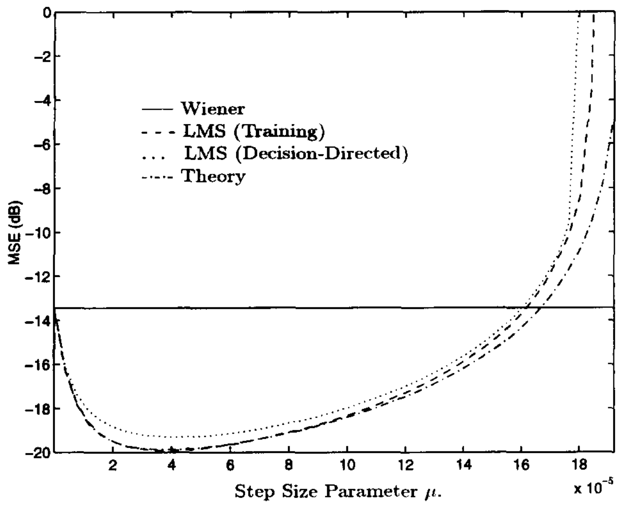

Figure 2. LMS MSE as a function of μ for a sinusoidal interference, with $L = 51$, SNR = 25dB, and SIR = -20dB.

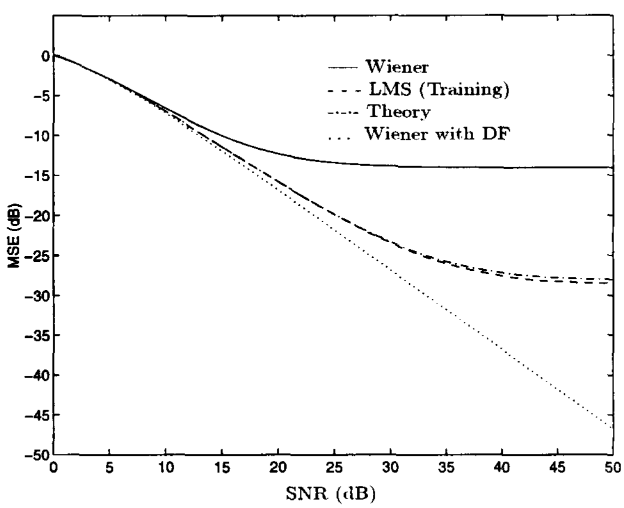

Figure 3. LMS MSE as a function of SNR for a sinusoidal interference, using optimum step size parameter μ_{opt}, L = 51, and SIR = -20dB.

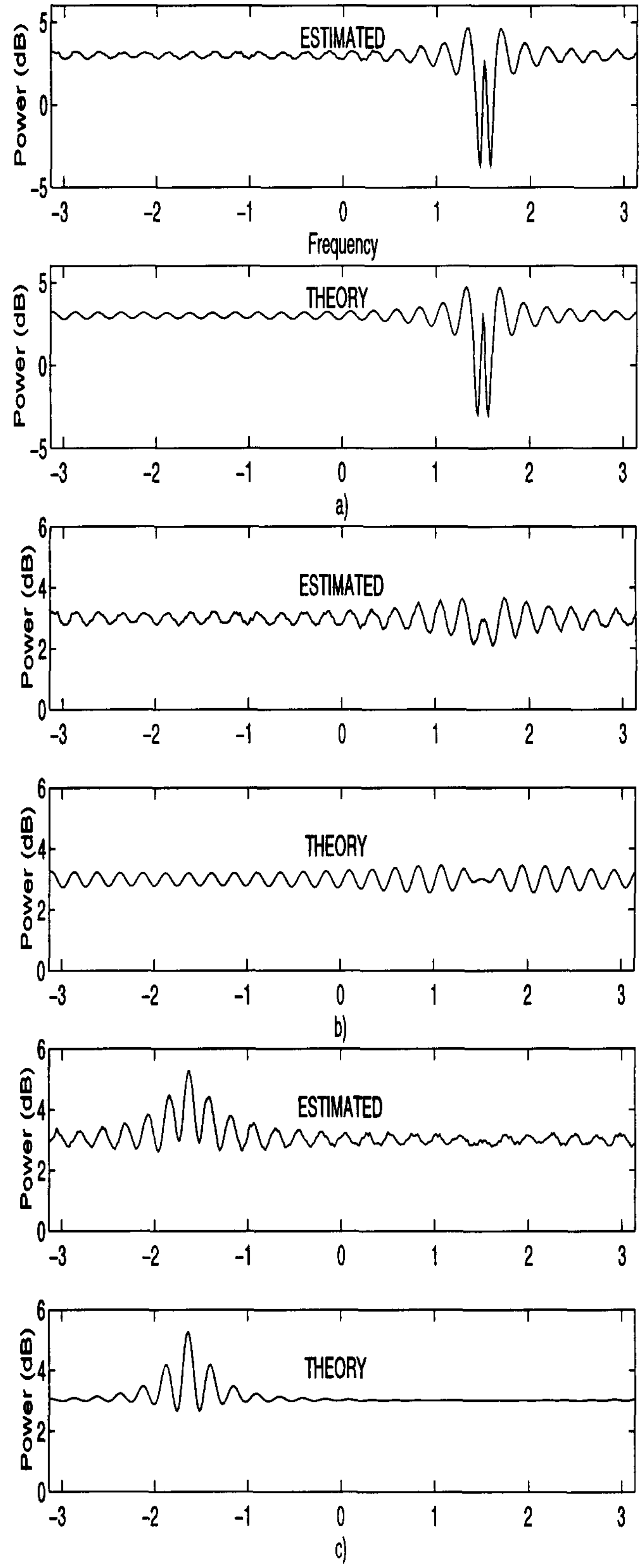

Figure 4. Estimated and theoretical power spectral densities of LMS output process $y(k)$ for a) $\mu = 2.50 \times 10^{-6}$, b) $\mu = 3.78 \times 10^{-5}$, c) $\mu = 1.76 \times 10^{-4}$, with $\omega_\Delta = 1.5$ and parameters as in Fig. 2.