

ABSTRACT

This paper is concerned with the efficient storage of the luminance parameters in a fractal code by means of vector quantization (VQ). For a given image block (range) the collage error as a function of the luminance parameters is a quadratic function with ellipsoid contour lines. We demonstrate how these functions should be used in an optimal codebook design algorithm leading to a non-standard VQ-scheme. In addition we present results and an evaluation of this approach. The analysis of the quadratic error functions also provides guidance for optimal scalar quantization.

1. INTRODUCTION

In fractal image compression [1, 2, 3] an image is partitioned into a set of image blocks called ranges. The ranges are matched with blocks taken from a codebook of filtered and subsampled image blocks ('domains') up to an affine transformation of intensity values. To find the optimal parameters of the affine transformation for a range R and a domain D , the least squares problem

$$(\hat{s}, \hat{o}) = \arg \min_{s, o \in \mathbb{R}} \|R - sD - o\mathbf{1}\|^2,$$

has to be solved, where $\mathbf{1}$ is the constant block with unit intensity at every pixel. The scaling parameter s and the offset o are called luminance parameters. The scaling coefficient s is clamped to $[-s_{max}, s_{max}]$, $0 \leq s_{max} < 1$, to ensure convergence in the decoding. The codebook block D_k that minimizes the collage error

$$E(D, R) = \|R - \hat{s}D - \hat{o}\mathbf{1}\|^2$$

yields the fractal code for range R consisting of the address k and the quantized coefficients. Since the quantization of $\hat{s}, \hat{o}$ increases the collage error, the codebook block which minimizes the collage error using quantized luminance parameters should be chosen.

The quadratic forms given by the collage error as a function of s, o (with D, R fixed),

$$E_{D,R}(s, o) = \|R - sD - o\mathbf{1}\|^2 = \langle D, D \rangle s^2 + 2\langle D, \mathbf{1} \rangle so + \langle \mathbf{1}, \mathbf{1} \rangle o^2 - 2\langle R, D \rangle s - 2\langle R, \mathbf{1} \rangle o + \langle R, R \rangle,$$

have ellipsoid contour lines (see Figure 1). Here, $\langle \cdot, \cdot \rangle$ denotes the vector dot product. For example, $\langle D, D \rangle$ is the

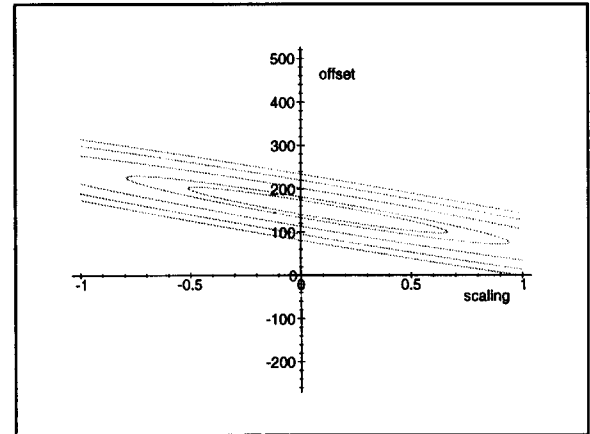


Figure 1. An example of the shape of the contour lines of $E(D, R)$ as a function of the luminance parameters.

sum of the square intensities in block D , and $\langle \mathbf{1}, \mathbf{1} \rangle$ equals the number of pixels in a block. The shape of the ellipses depends only on $\langle D, D \rangle s^2 + 2\langle D, \mathbf{1} \rangle so + \langle \mathbf{1}, \mathbf{1} \rangle o^2$. Thus, by determining the eigensystem of

$$\begin{pmatrix} \langle D, D \rangle & \langle D, \mathbf{1} \rangle \\ \langle D, \mathbf{1} \rangle & \langle \mathbf{1}, \mathbf{1} \rangle \end{pmatrix}$$

one gets the lengths and directions of the major and minor axes. When one considers the different orders of magnitude of the scaling and the offset parameter, such an analysis shows that the error is more sensitive to changes in the offset than to changes in the scaling component.

In Fisher's quadtree coder [3] a scalar quantization scheme is employed: first, $\hat{s}$ is quantized using a uniform quantizer with $2^{s_{bits}}$ levels for $[-s_{max}, s_{max}]$. Then the optimal offset for this quantized scaling coefficient is computed. This offset is quantized using a uniform quantizer for $[o_{min}(s), o_{max}(s)]$ with $2^{o_{bits}}$ levels, where

$$\begin{aligned} [o_{min}(s), o_{max}(s)] &= [-255s, 255] \text{ for } s \geq 0, \\ [o_{min}(s), o_{max}(s)] &= [0, 255(1-s)] \text{ for } s < 0. \end{aligned}$$

Since the s -value determines the interval of feasible optimal o -values, the accuracy of the offset quantization depends on the corresponding scaling coefficient. Figure 2 shows the grid of (s, o) -pairs used for quantization by

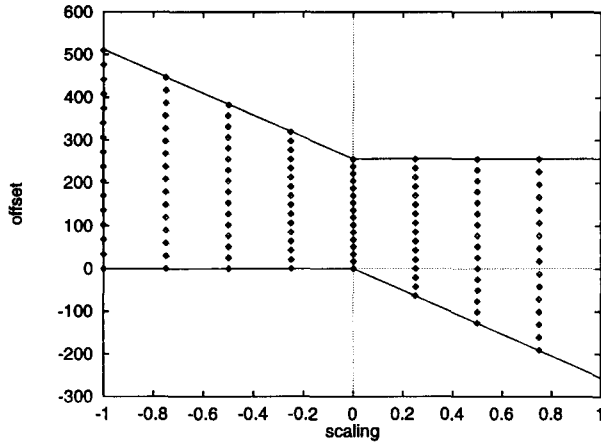


Figure 2. The (s, o) -pairs used in the quantization scheme of [3] ($s_{bits} = 3, o_{bits} = 4$), and the feasible region of (s, o) -pairs.

this scheme. Note that this scheme, at least theoretically, chooses neither the Euclidean nearest neighbor nor the optimal grid point.

In Figure 3 a distribution of optimal (s, o) -pairs is shown. Obviously, there is a certain structure that can be exploited using VQ-techniques. This idea was first proposed by Barthel et al. [4] and has been worked out in [6] and in this paper. In the following section we show the essential features of an optimal VQ-encoding in the context of fractal compression which significantly differs from standard VQ-techniques. In Section 3 we give experimental results and compare the VQ-scheme with modified scalar quantizers. A relation between the elliptic contour lines of the collage error functions and the optimal grid for scalar quantization is given in Section 4. Section 5 concludes the paper with a summary.

2. VQ-ENCODING OF LUMINANCE PARAMETERS

Let us denote the ranges and domains for a particular image encoding by $R_k, k = 1..n_R$, and $D_l, l = 1..n_D$, respectively. In addition, a codebook of n_C (s, o) -pairs is given. Then, the image encoding problem is to determine the functions

$$\begin{aligned} index : \{1, \dots, n_R\} &\rightarrow \{1, \dots, n_C\}, \\ address : \{1, \dots, n_R\} &\rightarrow \{1, \dots, n_D\} \end{aligned}$$

such that

$$\sum_{k=1}^{n_R} \|R_k - s_{index(k)} D_{address(k)} - o_{index(k)} \mathbf{1}\|^2 \quad (1)$$

has minimal value. Thus, for each range-domain pair (R, D) the codebook has to be searched for the entry which minimizes $\|R - sD - o\mathbf{1}\|^2$. Note that the best codebook entry is not necessarily the nearest one (in Euclidean sense) to the optimal $(\hat{s}, \hat{o})$. Moreover, it depends not only on $(\hat{s}, \hat{o})$, but also on the image data in R and D . Thus, our VQ-encoding problem is a non-standard one.

How do we get the luminance codebook? The codebook should be an integral part of the codec. Thus, a *good* codebook is a *small* set of (s, o) -pairs such that, when coding any

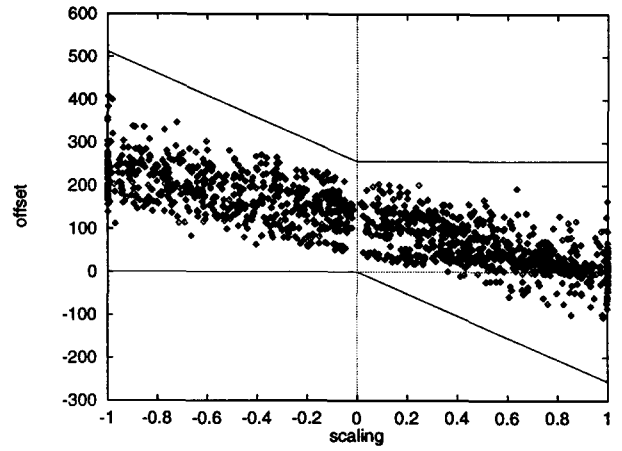


Figure 3. Optimal unquantized (s, o) -parameters of an encoding of Lenna 512×512 with 2663 ranges of sizes $4 \times 4, 8 \times 8, 16 \times 16$.

image, the optimum of (1) should be close to the theoretical limit

$$\sum_{k=1}^{n_R} \|R_k - \hat{s} D_{address(k)} - \hat{o} \mathbf{1}\|^2,$$

i.e., it should be close to the collage error that is gained using unquantized luminance parameters. To achieve this, we propose to take the quadratic forms q_k given by the optimal range-domain pairs of several images as a 'training sequence' and to solve the following optimization problem:

GIVEN: a set of functions

$$\begin{aligned} q_k(s, o) &= a_k s^2 + b_k s o + c_k o^2 + d_k s + e_k o + f_k, \\ k &= 1..n_q; \text{ the codebook size } n_C. \end{aligned}$$

GOAL: find n_C (s, o) -pairs to minimize

$$\sum_{k=1}^{n_q} q_k(s_{i(k)}, o_{i(k)}), \text{ where } i(k) \text{ is the index of the best codebook vector for function } q_k.$$

For this task we propose an iterative optimization procedure in the spirit of the LBG-algorithm. After initializing the codebook, the following two steps are iterated:

1. Clustering: For each q_k find the codebook entry (s_i, o_i) such that $q_k(s_i, o_i)$ is minimal. Put k into cluster C_i .
2. Codebook refinement: For each cluster C_i sum up the quadratics $q_k, k \in C_i$, yielding a quadratic $Q_i(s, o) = \sum_{k \in C_i} q_k(s, o)$ and replace (s_i, o_i) by the parameters $\arg \min_{(s, o) \in \mathbb{R}^2} Q_i(s, o)$.

Step 1 corresponds to the nearest neighbor condition of a standard VQ-design. Instead of simply searching the whole codebook every time, a more sophisticated strategy based on a bounding box criterion is used to reduce computation time. Step 2 is the appropriate modification of the standard centroid condition.

3. EXPERIMENTS

For our experiments we used a modified version [5] of the fractal coder described in [3]. This coder partitions the image using a quadtree scheme with a splitting criterion based on the variance weighted with the corresponding number of pixels in an image block, i.e., a range R is split when

$$\left\| R - \frac{\langle R, \mathbf{1} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} \mathbf{1} \right\|^2 > T,$$

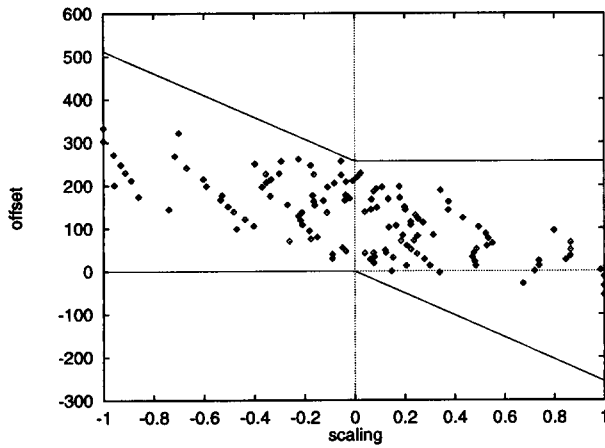


Figure 4. Codebook with 128 (s, o)-pairs.

where the threshold T is a parameter of the coding scheme. This splitting strategy does not degrade the performance in the rate distortion sense [5]. Moreover, the quadrees generated in this way do not depend on the collage error and quantization issues in contrast to Fisher's quadtree scheme in [3]. Thus, this approach is better suited for studying the quantization issues on hand here.

Several codebooks were designed for different thresholds T . A codebook containing 128 entries is depicted in Figure 4. Because of the ellipsoid form of the error functions, it is hard to tell from visual inspection whether those entries fit well to a distribution as depicted in Figure 3.

Tests were performed by coding Lenna 512×512 (not part of the codebook training sequence). As a reference, Table 1 gives the optimal collage error as well as the collage error of the scalar quantization scheme described in Section 1 for four different partitionings. The scalar quantizer uses eight bits per range for the luminance parameters. The selection of $s_{bits} = 3, o_{bits} = 5$ is experimentally determined to be the optimal choice for our coder. The results in Table 2 show that using the VQ-scheme savings of one bit per range can be achieved, i.e., only seven bits per range are used for the luminance parameters without degrading the collage error. This is in line with the fact that only about half of the feasible region is actually covered by optimal (s, o)-pairs (see Figure 3). However, almost the same savings can be expected by entropy coding of the s and o streams produced by the scalar quantizer. Indeed, the entropy for the o stream is about 3.9. The entropy of the index sequence of the VQ-scheme with a codebook of size 128 is about 6.5.

Another way of saving a bit per range is to restrict the values of o to half of the interval that is used in Fisher's scalar quantizer, i.e., to

$$[o_{min}(s) + \frac{l}{4}, o_{max}(s) - \frac{l}{4}], \quad l = o_{max}(s) - o_{min}(s).$$

The results of this modified scalar quantization are listed in the right column of Table 2. In this case the entropy of the o stream is 3.8. Table 3 lists compression results of VQ-encoding for codebook sizes 256 and 64. Of the three codebook sizes tested (64, 128, 256) the 7-bit codebook with 128 entries gives the best rate-distortion curve (but only by a very small margin).

number of ranges	optimal collage error PSNR (dB)	scalar quantization	
		compression ratio	$s_{bits} = 3$ $o_{bits} = 5$ PSNR (dB)
2632	31.03	36.2:1	30.42
1813	29.40	54.1:1	28.88
1516	28.47	65.8:1	28.00
1243	27.50	82.0:1	27.06

Table 1. Shown are the number of ranges of the range partitioning, the PSNR of the collage error using unquantized luminance parameters, and the compression ratio and collage error PSNR using the scalar quantizer described in Section 1.

number of ranges	comp. ratio	vector quantization 128 (s, o)-pairs	modified scalar quantization 128 (s, o)-pairs
2632	40.0:1	30.48	30.42
1813	56.8:1	28.93	28.87
1516	69.1:1	28.06	28.00
1243	86.2:1	27.13	27.06

Table 2. Compression results (collage error PSNR (dB)) using VQ of the luminance parameters with a codebook of size 128 and using the modified scalar quantizer.

number of ranges	vector quantization			
	256 (s, o)-pairs ratio	PSNR (dB)	64 (s, o)-pairs ratio	PSNR (dB)
2632	36.2:1	30.68	39.9:1	30.24
1813	54.1:1	29.11	59.7:1	28.70
1516	65.8:1	28.22	72.7:1	27.87
1243	82.0:1	27.27	90.9:1	26.92

Table 3. Compression results using VQ of the luminance parameters with codebook sizes 256 and 64.

When one takes into account the additional time needed for the codebook search in the VQ-encoding, the marginal gains with respect to scalar quantization appear to be too small to justify the extra time.

4. OPTIMAL BIT-ALLOCATION FOR SCALAR QUANTIZATION

The results of Section 3 indicate that scalar quantization (or Lattice-VQ) is the appropriate compromise between computing time, compression ratio, and quality. In this section we investigate the relation between the shape of the error functions and the optimal grid choice. For this purpose we will use the modified luminance transformation from [6]. Here, a range is approximated by

$$\lambda_{s,o}(D) = s \left(D - \frac{\langle D, 1 \rangle}{\langle 1, 1 \rangle} 1 \right) + s_0 \frac{\langle D, 1 \rangle}{\langle 1, 1 \rangle} 1 + o1,$$

where $s_0 \in [0, 1)$ is a fixed parameter of the method. For a discussion of this transformation see [6]. The shape of the error functions $\hat{E}_{D,R}(s, o) = \|R - \lambda_{s,o}(D)\|^2$ is determined by the quadratic

$$\left(\langle D, D \rangle - \frac{\langle D, 1 \rangle^2}{\langle 1, 1 \rangle} \right) s^2 + \langle 1, 1 \rangle o^2.$$

Therefore the major and minor axes of the elliptic contour lines are parallel to the coordinate axes. Generally, this is

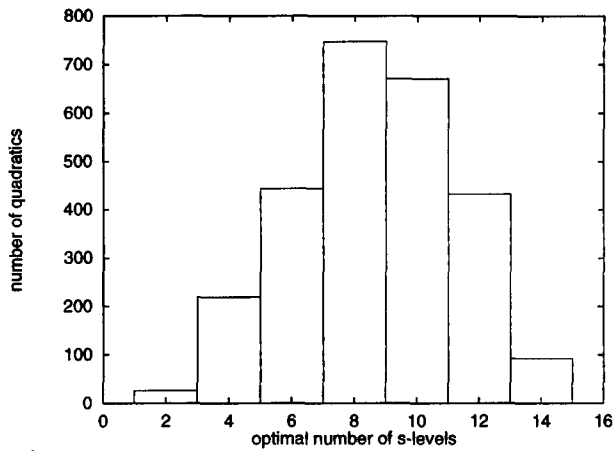


Figure 5. Distribution of optimal number of s -levels in a Lenna 512×512 encoding with 2662 ranges when 8 bits per range are used for (s, o) .

not the case for the affine mappings in the standard scheme.

By setting $s_0 = 0$ one obtains Øien's luminance transformation [7]. The quantization of the scaling parameters for this special case was done using a Lloyd-Max optimized quantizer [8]. This quantizer is based on the distribution of optimal s -values and does not take into account the actual error functions. In the following paragraph we will outline under somewhat idealized assumptions how the shapes of the error functions determine the optimal scalar quantization scheme.

For the sake of simplicity, let us assume that all error functions are of the shape $as^2 + co^2$ with the same coefficients a and c , and the minima are uniformly distributed in an area of size $l_1 \times l_2$. In practice, $l_1 = 2$ is the size of the range of s -values and l_2 is the size of the feasible range of o -values depending on s_0 . Which lattice with q points minimizes the expected quantization error? For step sizes $2\Delta_s, 2\Delta_o$ and a lattice point p all points in $(p - \Delta_s, p + \Delta_s] \times (p - \Delta_o, p + \Delta_o]$ are quantized to p . The expected quantization error $\mathcal{E}$ in this rectangle is

$$\mathcal{E}(\Delta_s, \Delta_o) = \int_{-\Delta_s}^{\Delta_s} \int_{-\Delta_o}^{\Delta_o} as^2 + co^2 dods = \frac{4}{3}(a\Delta_s^3\Delta_o + c\Delta_o^3\Delta_s).$$

Setting $v := l_1 l_2 / q$ and $\Delta_o := v / (4\Delta_s)$, one ends up with

$$\mathcal{E}(\Delta_s) = \frac{1}{3}a\Delta_s^2 v + \frac{1}{48}c\frac{v^3}{\Delta_s^2}.$$

The minimum of $\mathcal{E}(\Delta_s)$ is attained at the value $\Delta_s = 1/2 \cdot \sqrt[4]{cv^2/a}$. As expected, this leads to the ratio

$$\frac{\Delta_s}{\Delta_o} = \sqrt{\frac{c}{a}},$$

and $l_1 / \sqrt[4]{v^2 c / a}$ levels should be used to quantize s .

Of course, the above analysis is an idealization since the ratio between the lengths of the major and minor axes are not the same for all error functions. To obtain the optimal lattice in this case one also has to consider the distribution of shapes. Figure 5 lists a distribution of the

optimal number $2 / \sqrt[4]{v^2 c / a}$ of quantization levels for the scaling coefficient s , rounded to the nearest integer, when a lattice of 256 points is used. This distribution is obtained from the error functions of the optimal range-domain pairs of a Lenna 512×512 encoding using the modified luminance transformation. Obviously, no more than 16 levels are necessary for the quantization of s , and, when a power of two s -levels are used, eight s -levels would be optimal. This is in line with the experimental data (s -levels|collage error PSNR): (2|29.83), (4|30.36), (8|30.44), (16|30.27), (32|29.88), (64|25.83), (128|6.80).

It seems that the best coding results can be obtained with $s_0 = 0$ and an optimal quantization lattice, with combined DPCM/entropy-coding of the o -stream.

5. CONCLUSION

In this paper we have presented a new method of how VQ-encoding of the scaling-offset pairs in a fractal coding scheme should be done taking into account the ellipsoid forms of the collage error contour lines. This approach leads to a one bit per range saving compared to the standard scalar quantization. However, when compared to scalar quantization combined with entropy coding the benefits of the VQ-approach will be reduced to a marginal gain. This demonstrates that the structure of the distribution of optimal scaling-offset parameters is not strong enough to provide an improvement of fractal coding by VQ-encoding of the luminance parameters. Nevertheless, the analysis of the collage error functions leads to a better understanding of the bit allocation problem of scalar quantization of the luminance parameters.

REFERENCES

- [1] Jacquin, A. E., *Image coding based on a fractal theory of iterated contractive image transformations*, IEEE Trans. Image Processing 1 (1992) 18–30.
- [2] Barnsley, M., Hurd, L., *Fractal Image Compression*, AK Peters, Wellesley, 1993.
- [3] Fisher, Y., *Fractal Image Compression — Theory and Application*, Springer-Verlag, New York, 1994.
- [4] Barthel, K. U., Voyé, T., Noll, P., *Improved fractal image coding*, Proc. Picture Coding Symposium, Lausanne, 1993.
- [5] Saupe, D., Jakob, S., *Variance-based quadrees in fractal image compression*, Electronics Letters, to appear.
- [6] Barthel, K. U., *Festbildcodierung bei niedrigen Bitraten unter Verwendung fraktaler Methoden im Orts- und Frequenzbereich*, Dissertation, Institut für Fernmelde-technik, TU Berlin, Wissenschaft & Technik Verlag, Berlin, 1996.
- [7] Øien, G. E., Lepsoy, S., *Fractal-based image coding with fast decoder convergence*, Signal Processing 40 (1994) 105–117.
- [8] Øien, G. E., *Parameter quantization in fractal image coding*, Proc. IEEE International Conference on Image Processing (ICIP), Austin, Texas, 1994, 142–146.