

A SYSTEM/GRAPH THEORETICAL ANALYSIS OF ATTRACTOR CODERS

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ABSTRACT

This paper provides links between the young field of attractor coding and the well-established fields of systems theory and graph theory. Attractor decoders are modeled as linear systems whose stability is both necessary and sufficient for convergence of the decoder. This stability is dictated by the location of the eigenvalues of the sparse state transition matrix of the system. The relationship between these eigenvalues, spatial causality of the system, and the patterns of interdependency between signal elements (or image pixels) is investigated for several cases using concepts from graph and matrix theory.

1. INTRODUCTION

Fractal coders are coding systems that use the redundancy present in signals at different scales for compression. Most of these systems are based on the theory of *iterated function systems (IFS)* developed by Barnsley [1]. Jacquin proposed the first implementation of this theory for a completely automatic image coding system [2].

In this theory, a signal x is represented as the attractor of a contractive transformation $\mathcal{T}$. The encoder designs $\mathcal{T}$ and the decoder finds the attractor of $\mathcal{T}$ typically by applying $\mathcal{T}$ to an arbitrary initial signal x_0 repeatedly, generating a sequence of signals (x_n) that converge to an approximation of x . The design of $\mathcal{T}$ in the encoder is based on minimizing the distance $\varepsilon_E = d(x, \mathcal{T}(x))$. However, the goal of the system is to minimize $\varepsilon_D = d(x, \mathcal{T}^\infty(x))$. The Collage Theorem [1] provides an upper bound for ε_D in terms of ε_E . It has been found that even without the constraint of contractivity of $\mathcal{T}$, in many cases (x_n) still converges to a close approximation of x , sometimes even giving better coding performance. The concept of contractivity and the Collage Theorem has been further extended to *eventual contractivity* [3], which imposes a milder constraint on $\mathcal{T}$, but still is not a necessary condition for convergence. When (eventual) contractivity of $\mathcal{T}$ is not established, the Collage Theorem may not be used, and the convergence of (x_n) is not guaranteed. In the literature, little analysis has been provided on the behavior of the decoder in this case. On the other hand, it has been known that the structure of the interdependence of the signal elements in fractal coders closely affects the convergence and the error in the decoding sequence, but the exact relationship between these structures and the behavior of the decoder has not been established in the general case. Lundheim [4] noted that typical fractal

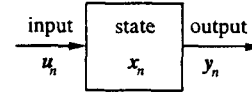


Figure 1: A discrete-time system

coders may be modeled with $\mathcal{T}$ being of the form

$$\forall x \in X, \quad \mathcal{T}(x) = Ax + B \quad (1)$$

in a vector space, and in such cases, the spectral radius of the linear part determines the convergence of the decoding sequence. He also found that when this linear part is a decimation matrix, the eigenvalues are related to cycles in the interdependence patterns of signal elements. Hürtgen and Simon [5] used the spectral radius of the linear part for analysis of convergence of some types of fractal coders.

In this paper we look at the attractor coding from the viewpoint of the systems theory, from which it becomes clear how the concept of the stability of a dynamical system provides a much more direct path to the convergence of attractor coders compared to the concept of contractivity which is being widely used in the literature of fractal coding. Using concepts from graph and matrix theory, we also see how eigenvalues of the state transition matrix are related to the structure of interdependence of the signal elements, and to the spatial causality of system, in a variety of cases. Finally, we take one step towards solving the general problem by providing a link between the coefficients of the characteristic equation of the state transition matrix and both the cycles in the flow graph of the system and the stability of the system, using theorems from graph and digital control theory.

2. MODELING ATTRACTOR CODING WITH LINEAR SYSTEMS

An alternative point of view for analyzing attractor coding systems is to look at them as *discrete-time systems* [6]. Consider a discrete-time system with input u_n , output y_n , and state x_n , as shown in Figure 1. The set of equations that describe the relationship between input, output, and state are called *dynamical equations*. Discrete-time systems are usually described by dynamical equations that are in the form

$$x_{n+1} = h(x_n, u_n, n) \quad (2)$$

$$y_n = g(x_n, u_n, n), \quad (3)$$

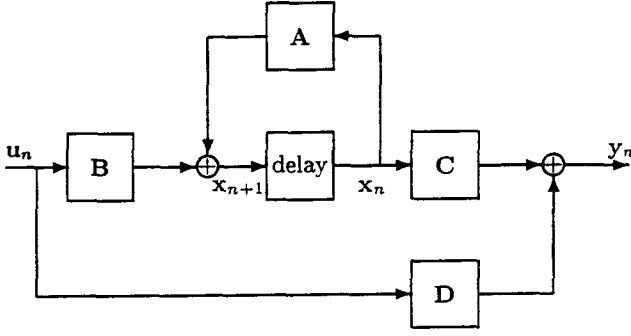


Figure 2: Block diagram of a linear time-invariant discrete-time system

where (2) is called the *state equation*, and (3) is called the *output equation*. If the system is *time-invariant*, these equations become

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{h}(\mathbf{x}_n, \mathbf{u}_n) \\ \mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n, \mathbf{u}_n). \end{aligned}$$

For linear time-invariant (LTI) systems, then the dynamical equations may be written as

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{B}\mathbf{u}_n \quad (4)$$

$$\mathbf{y}_n = \mathbf{C}\mathbf{x}_n + \mathbf{D}\mathbf{u}_n, \quad (5)$$

and may be represented by a block diagram as shown in Figure 2.

Note that the operator point of view described by (1) is equivalent to Equation (4) of a linear time-invariant system when $\mathbf{u}_n$ is constant. In other words, we may consider $x \in X$ as the state of a system which changes at each time step according to $\mathcal{T}$. The systems point of view provides for a more general class of attractor coders and may even be extended for systems that change continuously in time, as well as makes the connection between the young field of attractor coding and the well-established field of systems theory. Using the systems method, the output of the system may be defined to be different from (although a function of) the state of the system.

A comparison between (1) and (4) shows that the most common type of system being represented by attractor coders is, in fact, a linear system in which (1) the image is the *state* of the system, (2) the input is constant, (3) the initial image is the *initial state* of the system, and (4) the attractor is the *steady state* of the system. More interesting situations occur if we take the output equation (5) into account and define $\mathbf{y}_\infty$ as the final decoded signal in attractor coding. Then, if $\mathbf{C}$ is not identity, there may be state variables that are variable in time whose changes indirectly affect the output image. This has a strong relation to the concept of *hidden variables* defined in [1]. For example, $\mathbf{x}_n$ may be the coordinates of an evolving 3-D set and $\mathbf{y}$ a projection of that set into 2-D. The final decoded image is the projection of the attractor of the state into a 2-D plane.

3. STEADY STATE, ENCODER ERROR, AND DECODER ERROR

Now we concentrate our attention on Equation (4), assume a constant input $\mathbf{u}_n$ and represent $\mathbf{B}\mathbf{u}_n$ simply by $\mathbf{B}$:

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{B} \quad (6)$$

with the initial state $\mathbf{x}_0$. It can be shown that

$$\mathbf{x}_n - \mathbf{x}_\infty = \mathbf{A}^n(\mathbf{x}_0 - \mathbf{x}_\infty) \quad (7)$$

which shows an exponential convergence, if the state sequence $(\mathbf{x}_n)$ is convergent.

Let us assume that we apply the recursive formula of Equation (6) with $\mathbf{x}_0$ being the *original signal* (image) that is encoded. We define

$$\mathbf{e}_E \stackrel{\text{def}}{=} \mathbf{x}_0 - \mathbf{x}_1 \quad (8)$$

$$\mathbf{e}_D \stackrel{\text{def}}{=} \mathbf{x}_0 - \mathbf{x}_\infty \quad (9)$$

and

$$\varepsilon_E \stackrel{\text{def}}{=} \|\mathbf{e}_E\|$$

$$\varepsilon_D \stackrel{\text{def}}{=} \|\mathbf{e}_D\|$$

and call ε_E the encoding error (or the collage error), and call ε_D the decoding error. Then, it can be shown [7] that

$$\mathbf{e}_D = \mathbf{M}\mathbf{e}_E, \quad (10)$$

$$\mathbf{x}_\infty = \mathbf{M}\mathbf{B}, \quad (11)$$

$$\mathbf{x}_0 = \mathbf{M}(\mathbf{B} + \mathbf{e}_E), \quad (12)$$

where $\mathbf{M} \stackrel{\text{def}}{=} (\mathbf{I} - \mathbf{A})^{-1}$. Equation (10) provides an explicit relationship between the encoding error vector $\mathbf{e}_E$ and the decoding error vector $\mathbf{e}_D$, in contrast to the inequality of the Collage Theorem, and may be used to obtain the following bounds for ε_D ,

$$\varepsilon_D \geq \frac{\varepsilon_E}{1 + \|\mathbf{A}\|} \quad (13)$$

$$\varepsilon_D \leq \frac{\varepsilon_E}{1 - \|\mathbf{A}\|} \quad \text{if } \|\mathbf{A}\| < 1. \quad (14)$$

where (14) is the collage theorem in the discrete case. Equations (11) and (10) clearly show that the relationship between $\mathbf{e}_D$ and $\mathbf{e}_E$ is exactly the same relationship that $\mathbf{x}_\infty$ has with $\mathbf{B}$. Equations (11), (10) and (12) are summarized in Figure 3 and suggest that adding $\mathbf{e}_E$ to $\mathbf{B}$ changes the approximate decoder output $\mathbf{x}_\infty$ to the exact output $\mathbf{x}_0$. This may be interpreted that both $\mathbf{B}$ and $\mathbf{e}_E$ represent some type of residuals in the encoding process, with the difference that $\mathbf{B}$ is transmitted to the decoder, while $\mathbf{e}_E$ is typically not.

Using the notation $\mathbf{r} \stackrel{\text{def}}{=} \mathbf{B} + \mathbf{e}_E$, it can be shown that

$$\mathbf{x}_0 = \mathbf{A}\mathbf{x}_0 + \mathbf{r}. \quad (15)$$

$\mathbf{A}\mathbf{x}_0$ and $\mathbf{r}$ represent the *self-similarly* encoded component, and the *residual* component of $\mathbf{x}_0$, respectively, and $\mathbf{B}$ is the approximation of $\mathbf{r}$ which is sent to the decoder. If $\mathbf{r}$ was sent to the decoder without any error, then

$$\mathbf{r} = \mathbf{B} \Rightarrow \mathbf{e}_E = 0 \Rightarrow \mathbf{e}_D = 0 \Rightarrow \mathbf{x}_0 = \mathbf{x}_\infty$$

or using (15), we would have $\mathbf{x}_0 = \mathbf{M}\mathbf{r}$. This provides new insight into the encoding process:

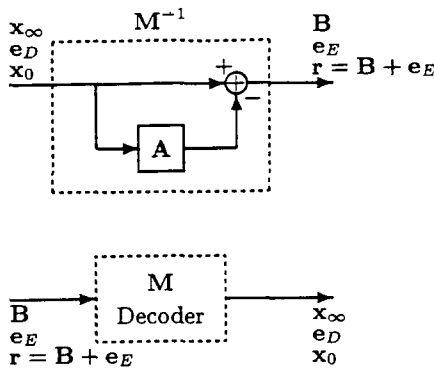


Figure 3: Block diagram representing the relation between B , x_∞ , e_E , e_D , r , and x_0

1. e_E is the error in transmitting the residual $r = x_0 - Ax_0$ to the decoder causing an error e_D at the decoder.
2. By transmitting B to the decoder, the encoder is practically encoding the residual of the image after removing the self-similar component of the image. Although this encoding is usually done by a simplistic method of sending the average value of the residual over each range block, more advanced techniques may be used for better encoding of this residual.
3. In order to make a lossless coder/decoder, one obvious method is for the encoder to decode the image, find e_D , and send it to the decoder in addition to the code for the fractal transform $\mathcal{T}$. However, (12) suggests the alternative approach of the encoder sending e_E rather than e_D to the decoder. The decoder may then subtract e_E from B and begin decoding.

If λ is an eigenvalue of A , from the definition of M it can be shown that $1/(1 - \lambda)$ is an eigenvalue of M and if λ is close to 1, the corresponding eigenvalue of M becomes large and any component of e_E along its eigenvector can cause a corresponding large component in e_D causing large decoding error.

4. STABILITY, CONTRACTIVITY, AND CONVERGENCE

The notions of contractivity and eventual contractivity are the main basis for analysis on convergence of attractor decoders in the literature on fractal coding. However, from the systems point of view, the convergence of the sequence (x_n) is extensively studied in terms of the *stability* of its generating system [6]. The system described in (4) and (5) is stable iff the eigenvalues of A have magnitudes less than 1, i.e., are located within the unit circle [6]. If A is an $N \times N$ matrix, and $\lambda_1, \lambda_2, \dots, \lambda_N$ are its eigenvalues, then

$$\rho(A) = \max_{1 \leq i \leq N} |\lambda_i|$$

is called the *spectral radius* of A . Hence, the stability of the system of (4) and (5) may be expressed as $\rho(A) < 1$.

In the literature, the rate of convergence of attractor decoders is typically analyzed in terms of the contractivity factor of the operator $\mathcal{T}$. For the linear time-invariant case, this is reduced to $\|A\|$. However, in this work, we propose that using the eigenvalues of A provides a more powerful tool for analysis of convergence in attractor decoders. However, computing the norm of A for some norms is easier than computing the eigenvalues. In the next section, we will investigate methods for computing λ 's using flow graphs of the matrix A .

The relationship between convergence of the sequence (x_n) and contractivity of $\mathcal{T}$ can be written as

$$\boxed{\mathcal{T} \text{ contractive}} \Rightarrow \boxed{\mathcal{T} \text{ eventually contractive}} \Rightarrow \boxed{\forall x, (\mathcal{T}^n(x)) \text{ convergent.}}$$

However, the relationship between convergence of (x_n) and stability of the discrete-time system is

$$\text{stable system} \iff \forall x, (\mathcal{T}^n(x)) \text{ convergent.}$$

One difficulty with using contractivity for analysis of convergence of (x_n) is that contractivity is based on $\|A\| < 1$ which depends on the definition of the norm being used. However, convergence of (x_n) is independent of the metric being used. There are A 's for which some norms are greater than 1 and some are less than 1. Although $\|A\| < 1$ is a sufficient condition for convergence, the reverse is not true. And, also, small $\|A\|$ guarantees a fast convergence, but again, in general, the reverse is not true. On the contrary, $\rho(A) < 1$ is both a necessary and sufficient condition for convergence and $\rho(A)$ directly dictates the speed of convergence in the long run as can be expected from (7).

5. GRAPH THEORETICAL APPROACH

5.1. Eigenvalues

In fractal image coding, image pixels are encoded by approximating them with other pixels in the same image. The structure of matrix A is determined by the pattern and level of interdependence of image pixels. However, each image pixel is typically dependent on only a small number of other pixels, and hence, the matrix A is very sparse. These dependencies may be better analyzed if represented by a flow graph. In such a representation, pixels are represented by vertices (nodes) and their dependencies by weighted arcs (edges). The resulting flow graph also provides a representation of matrix A . The eigenvalues of matrix A , and its stability, can be determined when the flow graph A is of any the following forms [7].

- single-path flow graph
- acyclic flow graph
- single-cycle flow graph
- multiple-component flow graph
- not-strongly-connected flow graph
- special cases of flow graphs with touching cycles

In general, touching cycles in the flow graph significantly complicate the eigenvalue problem. And in the general case

the relation between the spectral radius of $\mathbf{A}$ and these cycles seems to be unknown. However, using concepts from digital control theory we can establish links between these two. In terms of stability, there are methods for determining stability of a system from coefficient of its characteristic equation without solving the characteristic equation [6, Section 8-6][8, Section 6-3]. On the other hand, from a theorem in graph theory, it can be shown [7] that the coefficients of the characteristic equation have graph theoretical interpretations in terms of directed cycles of the Coates graph of matrix $\mathbf{A}$ [9, pages 206–210].

5.2. Spatial Causality

We call a system described by (4) and (5) *spatially causal* iff some permutation of $\mathbf{A}$ has zero elements, on and above the diagonal, i.e., is lower triangular with zero diagonal elements, or equivalently, if the flow graph of $\mathbf{A}$ is acyclic. We also call a system *spatially semicausal* if some permutation of $\mathbf{A}$ is lower triangular. This happens iff the only cycles in the flow graph of $\mathbf{A}$ are self-loops.

The concept of spatial causality in fractal coding has been addressed in [10]. It can be proven [7] that that spatial-causality is a sufficient condition for finite-iteration convergence. The concept of *Jordan canonical form* may be used to prove that any system may be transformed into a semicausal system by a proper change of basis [7]. If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A}$ can be transformed into a causal system whose flow graph is made up of only single-path components. If we further require that the basis be orthonormal in $\mathcal{C}^N$, we use *Schur's Theorem* [11] may be used [7] to prove that

Theorem: If the system $\mathbf{x}_{n+1} = \mathbf{A}_{N \times N} \mathbf{x}_n + \mathbf{B}_{N \times 1}$ reaches its steady-state in M iterations for any initial state $\mathbf{x}_0$, then (1) all the eigenvalues of $\mathbf{A}$ are zero, (2) the system also reaches its steady-state in N iterations (nontrivial when $N < M$), and (3) there is a basis in $\mathcal{C}^N$ such that the representation of $\mathbf{A}$ in that basis has a graph made of only directed path(s) with weights 1, and (4) there is an orthonormal basis in $\mathbf{A}$ such that the representation of $\mathbf{A}$ in that basis has an acyclic graph.

However, it can be shown by example that spatial causality is not a necessary condition for finite-iteration convergence.

6. CONCLUSIONS

In this paper, attractor coders are studied as discrete-time systems from the viewpoint of control systems theory. In light of this theory, the relationships between the decoded image, encoder error, and decoder error are investigated. It is also shown that the concept of stability of the discrete-time systems provides a more direct path to analyzing convergence in the attractor decoders. In contrast to contractivity, stability of the decoder is both necessary and sufficient for convergence in the decoder. To analyze the stability of these systems, a graph theoretical approach is used for evaluating the eigenvalues of the state transition matrix. At last, the effect of spatial causality of these systems on the convergence of the decoder is studied.

7. ACKNOWLEDGEMENT

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8. REFERENCES

- [1] M. F. Barnsley, *Fractals Everywhere*. San Diego, CA: Academic Press, Inc., 1988.
- [2] A. E. Jacquin, "A novel fractal block-coding technique for digital images," in *Proceedings of IEEE ICASSP-90*, (Albuquerque, NM), pp. 2225–2228, Apr. 3–6, 1990.
- [3] Y. Fisher and A. Lawrence, "Fractal image encoding," tech. rep., Netrologic, Inc., Nov. 17, 1990. SBIR Phase I, Final Report.
- [4] L. M. Lundheim, *Fractal Signal Modelling for Source Coding*. PhD thesis, The Norwegian Institute of Technology, Trondheim, Norway, Sept. 1992.
- [5] B. Hürtgen and S. F. Simon, "On the problem of convergence in fractal coding schemes," in *Proceedings of IEEE International Conference on Image Processing*, vol. 3, (Austin, TX), pp. 103–106, Nov. 13–16, 1994.
- [6] C.-T. Chen, *Linear System Theory and Design*. New York: Holt, Rinehart and Winston, Inc., 1984.
- [7] M. Gharavi-Alkhansari, *Fractal-Based Image and Video Coding Using Matching Pursuit*. PhD thesis, University of Illinois at Urbana-Champaign, Urbana, IL, 1997. To appear.
- [8] C. H. Houppis and G. B. Lamont, *Digital Control Systems: Theory, Hardware, Software*. New York: McGraw-Hill Book Company, 1985.
- [9] W.-K. Chen, *Applied Graph Theory*. New York: North-Holland Publishing Company, 1976.
- [10] M. Gharavi-Alkhansari and T. S. Huang, "Fractal-based techniques for a generalized image coding method," in *Proceedings of IEEE International Conference on Image Processing*, vol. 3, (Austin, TX), pp. 122–126, Nov. 13–16, 1994.
- [11] S. J. Leon, *Linear Algebra with Applications*. New York: Macmillan Publishing Company, 3rd ed., 1990.