THEORY AND DESIGN OF MULTIDIMENSIONAL TWO-CHANNEL NEAR-PERFECT-RECONSTRUCTION MODULATED FILTER BANKS *

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ABSTRACT

This article addresses the problem of designing two-channel near-perfect-reconstruction filter banks over multidimensional lattices. First, a cosine-modulated filter structure having arbitrary spatial shift and phase parameters is considered. The use of this structure leads to many possible two-channel multirate systems. The perfect reconstruction conditions are studied and appropriate choices for the parameters of the cosine-modulated structure are obtained. A simple but efficient unconstrained procedure for designing (possibly linear phase) near-perfect-reconstruction filter banks having good frequency responses (with arbitrary shapes) is proposed. A 2D design example is presented. The filter banks obtained can be used in transmultiplexers as well as in subband coders.

1. INTRODUCTION

Since the beginning of multirate theory, the two-channel case has played an important role in perfect reconstruction (PR) systems. Usually, one prototype filter (low-pass) is considered in the design and the other filters of the bank are obtained by *space reversal* and/or changes in the sign of some coefficients of that prototype [1, 2, 3]. In [1], the crosstalk or alias is zero but still a constrained optimization procedure is performed in order to achieve PR. In [4], non-separable filters are obtained from 1D filter banks but arbitrary shapes in the frequency domain cannot be obtained.

In this article, we improve several theoretical and design aspects of multidimensional two-channel PR filter banks. First, we propose a more general structure for the filters which leads to more system configurations (a wider choice of modulating frequencies). We show how to choose appropriate values of the parameters of such a structure leading to zero crosstalk (or alias) and good in-band recovery of the signals. We also show how to use two prototypes to add freedom while maintaining zero crosstalk (alias). An unconstrained procedure, based on the N-step Newton method, is proposed to design near-perfect-reconstruction (NPR) filter banks. These filters can have linear phase (or other symmetries such as quadrantal or octantal) and are optimized for good frequency responses. A 2D design example is presented which illustrates how the method can be used to easily design NPR filter banks having arbitrary shapes in the frequency domain.

In the context of this article, we are interested in both transmultiplexing and subband coding. Although we will present the theory from a transmultiplexing [5, 6] point of view, it applies to subband coding in a straightforward manner (since they are dual problems to each other). Results from lattice theory [7] will be used extensively in this article and the reader is assumed to be acquainted with this theory.

2. COSINE-MODULATED FILTER BANK STRUCTURE

Let $s_1(\mathbf{x})$ and $s_2(\mathbf{x})$ be two signals defined over a lattice $\Lambda \subset \Gamma$ such that[1] $\frac{d(\Gamma)}{d(\Lambda)} = 2$. We want to transmultiplex $s_1(\mathbf{x})$ and $s_2(\mathbf{x})$ over Γ. For this, we divide, in the frequency domain, a unit cell of Γ^* into two congruent bands, each occupied by a specific signal and congruent to a unit cell of Λ^*. The centers of these bands, occupied by $S_1(\mathbf{f})$ and $S_2(\mathbf{f})$, are $\mathbf{f}_{c_1}$ and $\mathbf{f}_{c_2}$ respectively (with $\mathbf{f}_{c_1} - \mathbf{f}_{c_2} \in \Lambda^* \setminus \Gamma^*$). The two-channel transmultiplexing system is illustrated in fig. 1 (a subband system is obtained by exchanging the order of the encoder and decoder). We consider for now the use of one low-pass prototype filter $g(\mathbf{x})$. The proposed cosine-modulated structure for the filters defined over Γ is:

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[1] A basis vector of Γ (in a certain basis) has been doubled and the others kept unchanged to form Λ.

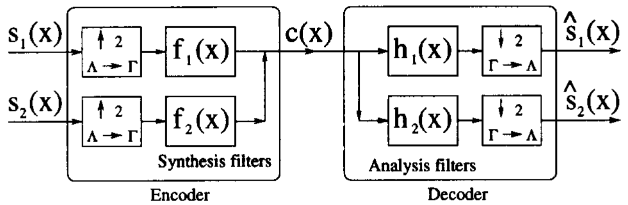

Figure 1: Two-channel multi-D transmultiplexer.

$$\begin{array}{ll} b_i(\mathbf{x}) & = g(\mathbf{x}-\mathbf{d}_i) \\ p_i(\mathbf{x}) & = g(\mathbf{x}-\mathbf{e}_i) \\ f_i(\mathbf{x}) & = b_i(\mathbf{x})\cos[2\pi\mathbf{f}_{c_i}^T(\mathbf{x}-\mathbf{d}_i)-\phi_i] \\ h_i(\mathbf{x}) & = p_i(\mathbf{x})\cos[2\pi\mathbf{f}_{c_i}^T(\mathbf{x}-\mathbf{e}_i)-\psi_i] \end{array} \quad (1)$$

with $\mathbf{x}, \mathbf{d}_i, \mathbf{e}_i \in \Gamma$, $\phi_i, \psi_i \in \mathbb{R}$, and $i \in \{1,2\}$.

The introduction of spatial shifts $\mathbf{d}_i$ and $\mathbf{e}_i$ and phases ϕ_i and ψ_i leads to some freedom allowing both crosstalk elimination and good in-band recovery.

As in [5], we choose[2] $\mathbf{f}_{c_i} \in \frac{1}{2}\Lambda^* \; \forall i$. The modulating frequencies of the filters are chosen as (for $i \in \{1,2\}$):

$$\mathbf{f}_{c_i} = \mathbf{q}_i + \mathbf{l}, \text{ with } \Lambda^* = \bigcup_{l=1}^{2}(\Gamma^* + \mathbf{q}_l), \; \mathbf{l} \in \tfrac{1}{2}\Lambda^*. \quad (2)$$

This leads to 2^D (the number of cosets in D dimensions of Λ^* in $\frac{1}{2}\Lambda^*$) possible systems, i.e. combinations of modulating frequencies, depending on vector $\mathbf{l}$. Since $\Lambda \subset \Gamma$ and $\frac{\mathsf{d}(\Gamma)}{\mathsf{d}(\Lambda)} = 2$, it follows that:

$$2\Gamma \subseteq \Lambda \Rightarrow \Lambda^* \subseteq \frac{1}{2}\Gamma^* \Rightarrow \mathbf{q}_i \in \frac{1}{2}\Gamma^*, \quad \text{since } \mathbf{q}_i \in \Lambda^*, \forall i$$

This implies that

$$\mathbf{f}_{c_i} = (\mathbf{q}_i + \mathbf{l}) \in \frac{1}{2}\Gamma^* \text{ iff } \mathbf{l} \in \frac{1}{2}\Gamma^*. \quad (3)$$

3. PERFECT RECONSTRUCTION CONDITIONS

The perfect reconstruction conditions are studied from two points of view: crosstalk elimination and in-band recovery of each signal. For both points of view two cases must be studied: $\mathbf{l} \in \frac{1}{2}\Gamma^*$ and $\mathbf{l} \notin \frac{1}{2}\Gamma^*$. For both cases, we show that crosstalk elimination is possible and we give explicit parameter conditions (i.e. conditions on $\mathbf{d}_i$, $\mathbf{e}_i$, ϕ_i and ψ_i) yielding such a result. We also give parameter conditions permitting in-band recovery. These values don't automatically yield perfect reconstruction although crosstalk is zero. The same situation arises in [1]. Optimization procedures, which will be described in section 5, must be used in order to obtain NPR.

First, we write the expressions of equation (1) in the frequency domain. Using Fourier transform properties [7] and the identity $\cos(\alpha) = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ it follows easily for $i \in \{1,2\}$ that:

$$\begin{array}{l} F_i(\mathbf{f}) = \frac{1}{2}e^{-j2\pi\mathbf{f}^T\mathbf{d}_i}\left[e^{-j\phi_i}G(\mathbf{f}-\mathbf{f}_{c_i}) + e^{j\phi_i}G(\mathbf{f}+\mathbf{f}_{c_i})\right] \\ H_i(\mathbf{f}) = \frac{1}{2}e^{-j2\pi\mathbf{f}^T\mathbf{e}_i}\left[e^{-j\psi_i}G(\mathbf{f}-\mathbf{f}_{c_i}) + e^{j\psi_i}G(\mathbf{f}+\mathbf{f}_{c_i})\right] \end{array} \quad (4)$$

Throughout this section, we will study the transfer function $T_{im}(\mathbf{f})$ from signal i to signal m defined as (from figure 1 and lattice theory results [7]):

$$\begin{array}{ll} T_{im}(\mathbf{f}) & = [F_i(\mathbf{f})H_m(\mathbf{f})] \downarrow \Lambda \\ & = \frac{1}{2}\sum_{l=1}^{2} F_i(\mathbf{f}-\mathbf{q}_l)H_m(\mathbf{f}-\mathbf{q}_l) \end{array} \quad (5)$$

3.1. Crosstalk elimination (case $\mathbf{l} \in \frac{1}{2}\Gamma^*$)

Since from (3), $2\mathbf{f}_{c_i} \in \Gamma^*$ and since $G(\mathbf{f})$ is Γ^*-periodic, it follows that $G(\mathbf{f}) = G(\mathbf{f} + 2\mathbf{f}_{c_i}), \forall i$. Using (2) and (4), we can write (5), for $i \neq m$, as [6]:

$$T_{im}(\mathbf{f}) = \tfrac{1}{2}\cos(\phi_i)\cos(\psi_m)G(\mathbf{f}-\mathbf{l})G(\mathbf{f}-\mathbf{f}_{c_1}-\mathbf{f}_{c_2} + \mathbf{l})e^{-j2\pi\mathbf{f}^T(\mathbf{d}_i+\mathbf{e}_m)}\left[e^{j2\pi\mathbf{q}_1^T(\mathbf{d}_i+\mathbf{e}_m)} + e^{j2\pi\mathbf{q}_2^T(\mathbf{d}_i+\mathbf{e}_m)}\right]$$

To eliminate the crosstalk, we can't choose $\cos(\phi_i) = 0$ or $\cos(\psi_m) = 0$ since the recovery of each signal would not be possible then (see subsection 3.3). We should instead make the term in brackets equal to zero:

$$\begin{array}{lrl} & e^{j2\pi\mathbf{q}_1^T(\mathbf{d}_i+\mathbf{e}_m)} + e^{j2\pi\mathbf{q}_2^T(\mathbf{d}_i+\mathbf{e}_m)} & = 0 \\ \Rightarrow & e^{j2\pi(\mathbf{f}_{c_1}-\mathbf{f}_{c_2})^T(\mathbf{d}_i+\mathbf{e}_m)} & = e^{j\pi} \\ \Rightarrow & (\mathbf{f}_{c_1}-\mathbf{f}_{c_2})^T(\mathbf{d}_i+\mathbf{e}_m) & = \frac{(2k+1)}{2}, \; k \in \mathbb{Z} \end{array}$$

Since $\mathbf{f}_{c_1} - \mathbf{f}_{c_2} \in \Lambda^*$, we must choose $\mathbf{d}_i + \mathbf{e}_m \in \frac{1}{2}\Lambda \setminus \Lambda$ such that $2(\mathbf{f}_{c_1} - \mathbf{f}_{c_2})^T(\mathbf{d}_i + \mathbf{e}_m)$ is odd. Thus, **we completely eliminate the crosstalk** if we choose:

1. $\mathbf{d}_1 + \mathbf{e}_2 \in \frac{1}{2}\Lambda \setminus \Lambda$ with $2(\mathbf{f}_{c_1} - \mathbf{f}_{c_2})^T(\mathbf{d}_1 + \mathbf{e}_2)$ odd.
2. $\mathbf{d}_2 + \mathbf{e}_1 \in \frac{1}{2}\Lambda \setminus \Lambda$ with $2(\mathbf{f}_{c_1} - \mathbf{f}_{c_2})^T(\mathbf{d}_2 + \mathbf{e}_1)$ odd.

We can show that solutions always exist [6].

3.2. Crosstalk elimination (case $\mathbf{l} \notin \frac{1}{2}\Gamma^*$)

In this case, from (3), we have for $i \neq m$:

$$\mathbf{f}_{c_i} + \mathbf{f}_{c_m} = \underbrace{(\mathbf{q}_i + \mathbf{q}_m)}_{\notin\Gamma^* \Rightarrow \in \Lambda^*\setminus\Gamma^*} + \underbrace{2\mathbf{l}}_{\in\Lambda^*\setminus\Gamma^*} \in \Gamma^*$$

since there are only 2 cosets of Γ^* in Λ^*. Using this fact, we write (5) (after some manipulations [6]) as:

$$T_{im}(\mathbf{f}) = \tfrac{1}{4}\sum_{l=1}^{2} e^{-j2\pi(\mathbf{f}-\mathbf{q}_l)^T(\mathbf{d}_i+\mathbf{e}_m)}\Big[\cos(\phi_i+\psi_m)G(\mathbf{f} - \mathbf{f}_{c_i} - \mathbf{q}_l)G(\mathbf{f}+\mathbf{f}_{c_i}-\mathbf{q}_l) + \tfrac{1}{2}\Big(e^{-j(\phi_i-\psi_m)} + e^{j(\phi_i-\psi_m)}e^{j2\pi(2\mathbf{f}_{c_i})^T(\mathbf{d}_i+\mathbf{e}_m)}\Big)G^2(\mathbf{f}-\mathbf{f}_{c_i}-\mathbf{q}_l)\Big]$$

Thus, **to completely eliminate the crosstalk**, we can make the two terms in brackets equal to zero by setting[3] ($\forall i \neq m$):

[2] For lattice Λ having sampling matrix $\mathbf{V}$, we denote by $\frac{1}{K}\Lambda$ the lattice having sampling matrix $\frac{1}{K}\mathbf{V}$ (i.e. each basis vector has been divided by scalar K).

[3] In this case, $\mathbf{d}_i + \mathbf{e}_m$ must be such that $4\mathbf{f}_{c_i}^T(\mathbf{d}_i + \mathbf{e}_m)$ is odd to also satisfy conditions for in-band recovery.

1. $\phi_i + \psi_m = \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$
2. $\phi_i - \psi_m = -\frac{\pi}{2}\left[4\mathbf{f}_{c_i}^T(\mathbf{d}_i + \mathbf{e}_m) - 1\right] + l\pi, \quad l \in \mathbb{Z}$

3.3. In-band recovery (case $\mathbf{l} \in \frac{1}{2}\Gamma^*$)

In this case, expansion of (5) gives

$$T_{ii}(\mathbf{f}) = \frac{1}{4}\overbrace{[\cos(\phi_i - \psi_i) + \cos(\phi_i + \psi_i)]}^{2\cos(\phi_i)\cos(\psi_i)}\sum_{l=1}^{2} G^2(\mathbf{f} - \mathbf{f}_{c_i} - \mathbf{q}_l)e^{-j2\pi(\mathbf{f}-\mathbf{q}_l)^T(\mathbf{d}_i+\mathbf{e}_i)}$$

We want to obtain the same recovery expression for the two signals (i.e. to have $T_{11}(\mathbf{f}) = T_{22}(\mathbf{f})$). For this we set $\mathbf{d}_1 + \mathbf{e}_1 = \mathbf{d}_2 + \mathbf{e}_2 = \mathbf{c} \in \Lambda$. We then can obtain **uniform in-band recovery** if the following conditions are met:

1. $\phi_1 \neq \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$ and $\psi_1 \neq \frac{(2l+1)\pi}{2}$, $l \in \mathbb{Z}$.
   $\phi_2 \neq \frac{(2r+1)\pi}{2}$, $r \in \mathbb{Z}$ and $\psi_2 \neq \frac{(2s+1)\pi}{2}$, $s \in \mathbb{Z}$.
2. $\phi_1 + \psi_1 = \pm(\phi_2 + \psi_2)$
3. $\phi_1 - \psi_1 = \pm(\phi_2 - \psi_2)$
4. $\mathbf{d}_1 + \mathbf{e}_1 = \mathbf{d}_2 + \mathbf{e}_2 = \mathbf{c} \in \Lambda$.

3.4. In-band recovery (case $\mathbf{l} \notin \frac{1}{2}\Gamma^*$)

In this case, we have [6]:

$$T_{ii}(\mathbf{f}) = \frac{1}{8}\sum_{l=1}^{2} e^{-j2\pi(\mathbf{f}-\mathbf{q}_l)^T(\mathbf{d}_i+\mathbf{e}_i)}\Big[e^{-j(\phi_i+\psi_i)}G^2(\mathbf{f} - \mathbf{f}_{c_i} - \mathbf{q}_l) + e^{j(\phi_i+\psi_i)}G^2(\mathbf{f} + \mathbf{f}_{c_i} - \mathbf{q}_l) + 2\cos(\phi_i - \psi_i)G(\mathbf{f} - \mathbf{f}_{c_i} - \mathbf{q}_l)G(\mathbf{f} + \mathbf{f}_{c_i} - \mathbf{q}_l)\Big]$$

We can prove [6] that we obtain **uniform in-band recovery** if the same conditions as those of the previous subsection are met except for the second one which should be written as:

2. $\phi_1 + \psi_1 = -(\phi_2 + \psi_2)$

For a transmultiplexing system to possess good performance, we have to combine the conditions for crosstalk elimination and in-band recovery. Solutions to both sets of conditions are easily obtained. In the case $\mathbf{l} \in \frac{1}{2}\Gamma^*$, we can choose ϕ_i and ψ_i as multiples of π. In the case $\mathbf{l} \notin \frac{1}{2}\Gamma^*$, we must choose ϕ_i and ψ_i as odd multiples of $\frac{\pi}{4}$. In both cases, $\mathbf{d}_i$ and $\mathbf{e}_i$ can't be zero for all i. Note the important role these spatial shifts are playing in crosstalk elimination.

4. THE USE OF TWO PROTOTYPES

We now introduce the use of two prototype filters $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ by modifying in (1) the following:

$$\begin{aligned} b_1(\mathbf{x}) &= g_1(\mathbf{x} - \mathbf{d}_1) & b_2(\mathbf{x}) &= g_2(\mathbf{x} - \mathbf{d}_2) \\ p_1(\mathbf{x}) &= g_2(\mathbf{x} - \mathbf{e}_1) & p_2(\mathbf{x}) &= g_1(\mathbf{x} - \mathbf{e}_2) \end{aligned}$$

Since crosstalk expressions contain only one prototype at a time, the conditions presented in the previous section regarding crosstalk remain valid. Indeed, from (5), we see that $T_{12}(\mathbf{f})$ is function of $G_1(\mathbf{f})$ only and $T_{21}(\mathbf{f})$ is function of $G_2(\mathbf{f})$ only.

The advantage of using two prototypes is that, while zero crosstalk is maintained, the flexibility for in-band recovery is increased. Indeed, with one prototype, we want $n_{ii}(\mathbf{x}) = g(\mathbf{x}) * g(\mathbf{x})$ to be a Nyquist filter with respect to Λ; i.e. having the property that $n_{ii}(\mathbf{x}) = 0$ $\forall \mathbf{x} \in \Lambda \setminus \{0\}$. This is more constrained than designing a Nyquist filter $n_{ii}(\mathbf{x}) = g_1(\mathbf{x}) * g_2(\mathbf{x})$. In fact, we do get better results in the latter case.

5. DESIGN OF NEAR-PERFECT-RECONSTRUCTION TWO-CHANNEL FILTER BANKS

We now propose an unconstrained optimization procedure, based on the N-step Newton method, to design two-channel near-perfect-reconstruction filter banks. The procedure is carried out in two steps:

1. Optimize the frequency responses of the filters ($g(\mathbf{x})$ or $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$) using the procedure described in [5] (convex problem).
2. Add a perfect-reconstruction spatial domain error function[4] (just the recovery aspect has to be considered) to the above procedure and continue optimization.

We have designed a 2D filter bank using such a procedure. We designed cross-shaped frequency responses to show that the procedure works for any desired shape in the frequency domain. The sampling lattices for $i \in \{1, 2\}$ are as follows:

$$\Gamma = \text{LAT}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) \Rightarrow \Gamma^* = \text{LAT}\left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right)$$

$$\Lambda_i = \text{LAT}\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right) \Rightarrow \Lambda_i^* = \text{LAT}\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix}\right)$$

The modulating frequencies are chosen as:

$$\mathbf{f}_{c_1} = [0, 0]^T, \qquad \mathbf{f}_{c_2} = [\tfrac{1}{2}, \tfrac{1}{4}]^T$$

An appropriate choice of parameters is:

$$\begin{aligned} &\phi_1 = 0, \quad \psi_1 = 0, \quad \mathbf{d}_1 = [1, 0]^T, \quad \mathbf{e}_1 = -[1, 0]^T \\ &\phi_2 = 0, \quad \psi_2 = 0, \quad \mathbf{d}_2 = [0, 0]^T, \quad \mathbf{e}_2 = [0, 0]^T \end{aligned}$$

We designed filters with quadrantal symmetry having 84 independent coefficients (14×6) for analysis and synthesis. The normalized magnitude responses of analysis and synthesis filters for signal 1 are shown in figure 2 (a) and (b). Filters for signal 2 are just modulated versions of these filters. As we already mentioned,

[4] The objective is to have $t_{ii}(\mathbf{x}) = \delta(\mathbf{x})$, i.e. 1 if $\mathbf{x} = \mathbf{0}$ and 0 otherwise.

the crosstalk is canceled. The recovery obtained, using two prototypes, is excellent for video applications (NPR is achieved) and is shown in figure 2 (c).

6. CONCLUSIONS

In this article, we studied the problem of designing two-channel NPR modulated filter banks. We presented filter structures which allowed exactly zero crosstalk between signals (regardless of the order of the filters) and a good in-band recovery by proper choice of parameter values. The choice of these parameter values has been studied in detail. We saw that some freedom in the design can be obtained by using two prototype filters. We finally proposed a design procedure for arbitrary passbands that leads to good results as showed in a design example.

7. REFERENCES

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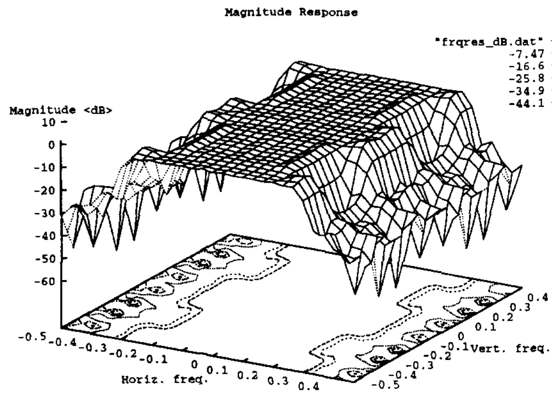
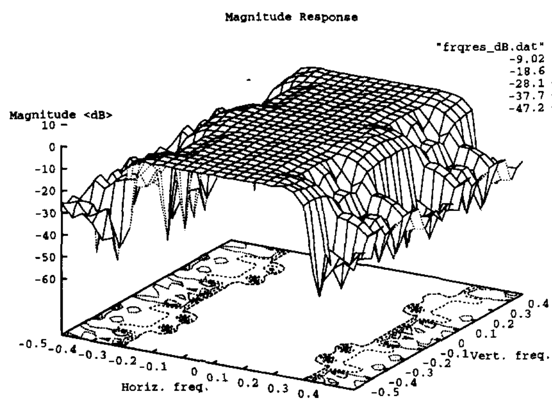
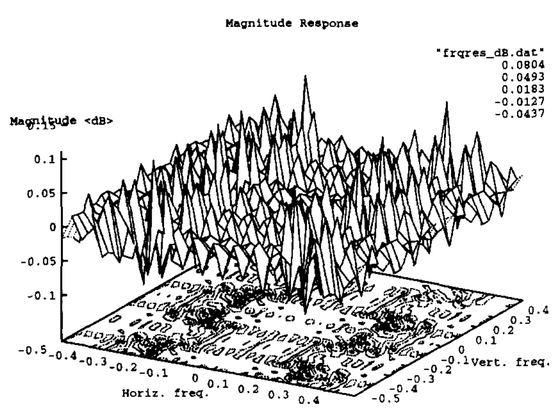


(a) Magnitude response of $F_1(\mathbf{f})$.



(b) Magnitude response of $H_1(\mathbf{f})$.



(c) Magnitude response of $T_{ii}(\mathbf{f}), i = 1, 2$.

Figure 2: Normalized magnitude responses (in dB) of designed filters ($F_1(\mathbf{f})$ and $H_1(\mathbf{f})$) and transfer functions ($T_{11}(\mathbf{f}) = T_{22}(\mathbf{f})$) of the 2D two-channel system.