COMPARATIVE STUDY OF IQML AND MODE FOR DIRECTION-OF-ARRIVAL ESTIMATION

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ABSTRACT

We present a comparative study of using the IQML (iterative quadratic maximum likelihood) algorithm and the MODE (method of direction estimation) algorithm for direction-of-arrival estimation with a uniform linear array. The consistent condition and the theoretical mean-squared error for the parameter estimates of IQML are presented. The computational complexities of both algorithms are also compared. We show that the frequency estimates obtained via MODE are asymptotically statistically efficient, while those obtained via IQML are almost always inconsistent and hence inefficient. We also show that the amount of computations required by IQML is usually much larger than that required by MODE, especially for low signal-to-noise ratio and large number of snapshots.

1. INTRODUCTION

The iterative quadratic maximum likelihood (IQML) algorithm [1, 2] is a popular algorithm for direction-of-arrival (DOA) estimation with a uniform linear array (ULA) and IQML is used to approximate the global minimizer of a deterministic maximum likelihood criterion. However, the direction estimates obtained by using IQML are almost always inconsistent.

In this paper, the consistent condition and the theoretical performance of the IQML algorithm are presented. We also compare the performance and the computational complexity of IQML with those of a similarly structured algorithm, the MODE (method of direction estimation) algorithm [3], which is an asymptotically statistically efficient estimator for the stochastic data model. We show that the direction estimates obtained via IQML are almost always inconsistent and hence inefficient and the bias dominates the mean-squared error (MSE) of the estimates asymptotically. We also show that the amount of computations required by IQML is usually much larger than that required by MODE, especially for low signal-to-noise ratio (SNR) and large number of snapshots.

2. PROBLEM FORMULATION

The problem of finding the DOAs of K narrow-band plane waves impinging on a ULA of M sensors can be reduced to that of estimating the spatial frequencies $\mathbf{f} = [\ f_1 \quad f_2 \quad \cdots \quad f_K\ ]^T$ in the following model:

$$\mathbf{y}(n) = \mathbf{A}\mathbf{x}(n) + \mathbf{e}(n), \quad n = 0, 1, \cdots, N-1, \tag{1}$$

where $\mathbf{y}(n) \in \mathcal{C}^{M\times 1}$ is the noisy observation vector, $\mathbf{x}(n) \in \mathcal{C}^{K\times 1}$ is the signal vector, $\mathbf{e}(n) \in \mathcal{C}^{M\times 1}$ is an unmeasurable noise process, N is the number of the data snapshots, and

$$\mathbf{A} = [\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_K\ ] \in \mathcal{C}^{M\times K}, \tag{2}$$

with

$$\mathbf{a}_k = [\ 1 \quad e^{j2\pi f_k} \quad \cdots \quad e^{j2\pi f_k(M-1)}\ ]^T \in \mathcal{C}^{M\times 1}, \quad k = 1, 2, \cdots, K. \tag{3}$$

Here $(\cdot)^T$ denotes the transpose. We assume that the signal, $\mathbf{x}(n)$, and the noise, $\mathbf{e}(n)$, are independent zero-mean complex Gaussian random processes with the following second-order moments:

$$\begin{array}{ll} \mathrm{E}\{\mathbf{x}(n)\mathbf{x}^H(l)\} = \mathbf{P}\delta_{n,l}, & \mathrm{E}\{\mathbf{x}(n)\mathbf{x}^T(l)\} = \mathbf{0}, \\ \mathrm{E}\{\mathbf{e}(n)\mathbf{e}^H(l)\} = \sigma^2\mathbf{I}\delta_{n,l}, & \mathrm{E}\{\mathbf{e}(n)\mathbf{e}^T(l)\} = \mathbf{0}, \end{array} \tag{4}$$

where $\mathrm{E}\{\cdot\}$ indicates the expectation, $(\cdot)^H$ denotes the complex conjugate transpose, $\mathbf{P} \in \mathcal{C}^{M\times M}$ is the unknown signal covariance matrix, $\mathbf{I} \in \mathcal{C}^{M\times M}$ is the identity matrix, $\delta_{n,l}$ represents the Kronecker delta, and σ^2 is the noise power.

3. CONSISTENT CONDITION OF IQML

Let $\{b_k\}_{k=0}^K$ be defined through the following polynomial:

$$p(z) \triangleq b_0 z^K + b_1 z^{K-1} + \cdots + b_K \triangleq b_0 \prod_{k=1}^{K}(z - e^{j2\pi f_k}), \tag{5}$$

$$\mathbf{B} = \begin{bmatrix} b_0 & & 0 \\ \vdots & \ddots & \\ b_K & & b_0 \\ & \ddots & \vdots \\ 0 & & b_K \end{bmatrix} \in \mathcal{C}^{M\times(M-K)}, \tag{6}$$

and

$$\mathrm{vec}(\mathbf{B}) \triangleq \bar{\mathbf{I}}\mathbf{b} = \bar{\mathbf{I}}\mathbf{W}\mathbf{p} \triangleq \mathbf{U}\mathbf{p}, \tag{7}$$

where $\mathbf{b} = [\ b_0 \quad b_1 \quad \cdots \quad b_K\ ]$, $\mathrm{vec}[\mathbf{X}]$ denotes the vector $[\ \mathbf{x}_1^T \quad \mathbf{x}_2^T \quad \cdots \quad \mathbf{x}_K^T\ ]^T$ with $\{\mathbf{x}_k\}_{k=1}^K$ being the columns of $\mathbf{X}$,

$$\bar{\mathbf{I}} = \left[\ \mathbf{I} \quad 0 \quad \cdots \quad 0 \ \vdots\ \cdots \ \vdots\ 0 \quad \cdots \quad 0 \quad \mathbf{I}\ \right]^T, \tag{8}$$

$\mathbf{W} \in \mathcal{C}^{(K+1)\times(K+1)}$ denoting a matrix made from 0, 1, $\pm j$, and $\mathbf{p}$ is a real-valued $(K+1)$-vector. Let $\mathbf{u}$ be a given constraint vector such that $\mathbf{u}^T\mathbf{u} = \mathbf{u}^T\mathbf{p} = 1$. We can prove that under the conjugate symmetry constraint [3]

$$b_k = b_{K-k}^*, \quad k = 0, 1, \cdots, K, \tag{9}$$

the estimates obtained with the IQML algorithm (after convergence, whenever this appears) are consistent only if [5]

$$\mathrm{Re}\left\{\mathbf{U}^H \mathrm{vec}\left[\mathbf{B}\left(\mathbf{B}^H\mathbf{B}\right)^{-1}\right]\right\} \sim \mathbf{u}, \tag{10}$$

where $\mathbf{x} \sim \mathbf{u}$ means that $\mathbf{x}$ is parallel to $\mathbf{u}$. Note that (10) depends on $\mathbf{p}$ only, that is, it depends on neither the signal covariance matrix $\mathbf{P}$ nor the noise power σ^2.

The condition on $\mathbf{p}$ in Equation (10) is almost never satisfied. More exactly, this condition imposes nontrivial constrains on the components of $\mathbf{p}$, which are only satisfied on zero-measure sets in the parameter space. To illustrate this fact, we consider a simple example in what follows.

Consider the usual choice of $\mathbf{u} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$. Also, let $K = 1$ and $M = 2$. The conjugate symmetry constraint in (9) can be written as:

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix} \begin{bmatrix} \mathrm{Re}(b_0) \\ \mathrm{Im}(b_0) \end{bmatrix} \triangleq \mathbf{W}\mathbf{p}. \tag{11}$$

Furthermore, for $M = 2$ and $K = 1$, we have $\mathbf{B} = \begin{bmatrix} b_0 & b_1 \end{bmatrix}^T$, which implies that $\mathbf{U} = \mathbf{W}$. Hence, from (10), we obtain:

$$\mathrm{Re}[j(b_1 - b_0)] = 2\mathrm{Im}(b_0) = 0, \tag{12}$$

which shows that the IQML is consistent in this case only if b_0 is *real-valued.* To translate this condition into one on the frequency parameters, note from (5) that $b_0^* + b_1^* e^{j2\pi f} = 0$, which is equivalent to $e^{j2\pi f} = -1$. For $f \in \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}$, the solution are $f = \pm 0.5$. It follows that the consistency condition in (10) is satisfied *if and only if* $f = \pm 0.5$ (this would correspond to end-fire positions).

4. THEORETICAL PERFORMANCE

We present below the asymptotic (for large N) statistical performance of IQML and MODE.

4.1. Performance of IQML

Let $\mathbf{V}$ be any $(K+1)\times K$ matrix such that $\begin{bmatrix} \mathbf{u} & \mathbf{V} \end{bmatrix}$ is an orthogonal matrix. To eliminate the constraint on $\mathbf{p}$, let $\mathbf{q}$ be a real-valued $K \times 1$ vector such that $\mathbf{p} = \mathbf{V}\mathbf{q} + \mathbf{u}$. Let $\mathbf{\Sigma} = \mathbf{V}^T\mathbf{U}^H\left[\bar{\mathbf{B}}(\bar{\mathbf{B}}^H\bar{\mathbf{B}})^{-1}\right]^T \otimes \mathbf{I}$ with $\otimes$ denoting the Kronecker matrix product and $\bar{\mathbf{B}}$ being defined similarly to $\mathbf{B}$ except that $\{b_k\}$ in $\mathbf{B}$ is replaced by its IQML estimate when $N \to \infty$, $\mathbf{\Phi} = \left\{\mathbf{r}_i^T \otimes \mathbf{r}_j\right\}_{i,j=1}^M$ with $\mathbf{r}_i$ denotes the ith column of $\mathbf{R} \triangleq \mathrm{E}\left\{\mathbf{y}(n)\mathbf{y}^H(n)\right\}$, and $\mathbf{\Gamma} = \mathbf{V}^T\bar{\mathbf{Q}}\mathbf{V} - 2\mathbf{V}^T\mathrm{Re}\{\mathbf{U}^H[(\bar{\mathbf{B}}^H\bar{\mathbf{B}}^{-T} \otimes \mathbf{R}\bar{\mathbf{B}}(\bar{\mathbf{B}}^H\bar{\mathbf{B}}^{-1}\bar{\mathbf{B}}^H]\mathbf{U}\}\mathbf{V}$ with $\mathbf{Q} = \mathrm{Re}\left\{\mathbf{U}^H\left[\left(\mathbf{B}^H\mathbf{B}\right)^{-T} \otimes \mathbf{R}\right]\mathbf{U}\right\}$. Let $\mathbf{D} = -\mathrm{Im}\{\mathbf{D}_1\mathbf{D}_2^{-1}\mathbf{D}_3\mathbf{W}\mathbf{V}\}$, where $\mathbf{D}_1 = \mathrm{diag}\{z_1^*, z_2^*, \cdots, z_K^*\}$ with $\{z_k\}$ denoting the zeros of the polynomial $p(z)$ in (5), $\mathbf{D}_2 = \mathrm{diag}\{\delta_1, \delta_2, \cdots, \delta_K\}$ with $\delta_k = \sum_{i=0}^K (K-i) z_k^{K-i-1} b_i$, and $\mathbf{D}_3$ is a Vandermond matrix obtained from $\{z_m\}_{m=1}^K$ that is rotated 90° clock-wise.

We can prove that the asymptotic MSE of $\hat{\mathbf{f}}$ obtained with IQML is:

$$\mathbf{MSE}\left(\hat{\mathbf{f}}\right) = \tfrac{1}{2N(2\pi)^2}\mathbf{D}\left[\mathbf{\Gamma}^{-1}\mathrm{Re}\left\{\mathbf{\Sigma\Phi\Sigma}^T + \mathbf{\Sigma}(\mathbf{R}^T \otimes \mathbf{R})\cdot \mathbf{\Sigma}^H\right\}\mathbf{\Gamma}^{-T} + (\bar{\mathbf{q}} - \mathbf{q})(\bar{\mathbf{q}} - \mathbf{q})^T\right]\mathbf{D}^T, \tag{13}$$

where $\bar{\mathbf{q}}$ is the estimate of $\mathbf{q}$ obtained by IQML when $N \to \infty$.

4.2. Performance of MODE

It has been shown in [3] that MODE is an asymptotically (for large N) statistically efficient estimator and the MSE of the estimates obtained via MODE asymptotically reaches the following Cramer-Rao bound (CRB):

$$\mathbf{CRB}\left(\hat{\mathbf{f}}\right) = \frac{\sigma^2}{2N}\left[\mathrm{Re}\{(\mathbf{D}_\mathbf{f}^H\mathbf{P}_\mathbf{A}^\perp\mathbf{D}_\mathbf{f}) \odot (\mathbf{S}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{S})^T\}\right], \tag{14}$$

where $\mathbf{P}_\mathbf{A}^\perp = \mathbf{I} - \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$, $\odot$ denotes the Hadamard (or Schur) product, i.e., element-wise matrix multiplication, and the kth column of $\mathbf{D}_\mathbf{f}$ is $\partial\mathbf{a}_k/\partial f_k, \quad k = 1, 2, \cdots, K$.

We note that $\mathbf{MSE}(\hat{\mathbf{f}})$ is different from $\mathbf{CRB}(\hat{\mathbf{f}})$. Furthermore, the difference between $\mathbf{MSE}(\hat{\mathbf{f}})$ and $\mathbf{CRB}(\hat{\mathbf{f}})$ becomes more significant as N increases since the IQML estimates are almost always biased.

5. COMPUTATIONAL COMPLEXITIES

5.1. Computations of IQML

An efficient implementation of the IQML algorithm for $N = 1$ is presented in [4]. The implementation can be readily extended to the case of $N > 1$, where the estimate of $\mathbf{p}$ is given by the following iterative procedure:

$$\hat{\mathbf{p}}^{(l+1)} = \min\mathbf{p}\left\{\mathbf{p}^H\mathrm{Re}\left\{\mathbf{W}^H\sum_{n=0}^{N-1}\left[\mathbf{Y}(n)^H(\mathbf{B}_l^H\mathbf{B}_l)^{-1}\cdot \mathbf{Y}(n)\right]\mathbf{W}\right\}\mathbf{p}\right\}, \tag{15}$$

where $\mathbf{B}_l$ is defined similarly to $\mathbf{B}$ except that $\mathbf{b}$ in $\mathbf{B}$ is replaced by $\hat{\mathbf{b}}_l = \mathbf{W}\hat{\mathbf{p}}^{(l)}$,

$$\mathbf{Y}(n) = \begin{bmatrix} y_{K+1}(n) & y_K(n) & \cdots & y_1(n) \\ y_{K+2}(n) & y_{K+1}(n) & \cdots & y_2(n) \\ \vdots & \vdots & \ddots & \vdots \\ y_M(n) & y_{M-1}(n) & \cdots & y_{N-K}(n) \end{bmatrix}, \tag{16}$$

with $y_m(n)$ being the mth element of $\mathbf{y}(n)$.

The idea of [4] is to reduce the amount of computations by utilizing the facts that $\mathbf{B}_l$ is a banded Toeplitz matrix and $\mathbf{B}_l^H\mathbf{B}_l$ is a banded Hermitian matrix.

The steps and the amount of computations required in each step of minimizing (15) are summarized as follows:

Step 1: Compute $\mathbf{C} = \mathbf{B}_l^H\mathbf{B}_l$. This step requires $O\left[\frac{1}{2}(K+1)(K+2)\right]$ flops.

Step 2: Compute the Cholesky decomposition $\mathbf{G}\mathbf{G}^H$ of $\mathbf{C}$. This step requires about $O\left[(M-K)\left(\frac{1}{2}K^2 + \frac{3}{2}K\right)\right]$ flops.

Step 3: Compute $\mathbf{Z}(n)=\mathbf{G}^{-1}\mathbf{Y}(n)$, $n=0,1,\cdots,N-1$. This step requires $O\left[(M-K)(K+1)^2N\right]$ flops.

Step 4: Compute $\mathbf{\Psi}(n)=\mathbf{Z}^H(n)\mathbf{Z}(n)$, $n=0,1,\cdots,N-1$. Since $\mathbf{Z}(n)$ is an $(M-K)\times(K+1)$ matrix, with the considerations of the Hermitian structure in this matrix and $n=0,\cdots,N-1$, we know that this step requires about $O\left[\frac{1}{2}(M-K)(K+2)(K+1)N\right]$ flops.

Step 5: Compute $\mathbf{\Omega}=\mathrm{Re}\left\{\mathbf{W}^H\left\{\sum_{n=0}^{N-1}\mathbf{\Psi}(n)\right\}\mathbf{W}\right\}$. This step requires about $O\left[2(K+1)^3+2(K+1)^2N\right]$ flops.

Step 6: Compute $\hat{\mathbf{p}}=\arg\min\mathbf{p}\left\{\mathbf{p}^T\mathbf{\Omega}\mathbf{p}\right\}$ such that $\mathbf{u}^T\mathbf{p}=1$. This step requires about $O\left[\frac{1}{3}K^3\right]$ flops.

Let L denote the number of iterations required by the IQML algorithm to achieve convergence. For $N\gg M\gg K$, which occurs often in angle estimation applications, IQML requires about $O\left[\frac{1}{2}(K+1)(3K+4)(M-K)NL\right]$ flops.

5.2. Computations of MODE

The MODE algorithm estimates $\mathbf{b}$ by

$$\hat{\mathbf{b}}=\arg\min_{\mathbf{b}}\mathrm{tr}\left[(\hat{\mathbf{S}}^H\mathbf{B})(\mathbf{B}^H\mathbf{B})^{-1}(\mathbf{B}^H\hat{\mathbf{S}})\hat{\tilde{\mathbf{\Lambda}}}\right],\tag{17}$$

where the columns in $\hat{\mathbf{S}}\in\mathcal{C}^{M\times\bar{K}}$ are the eigenvectors of $\hat{\mathbf{R}}=\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{y}(n)\mathbf{y}^H(n)$ that correspond to the $\bar{K}$ ($\bar{K}=\min\{N,\mathrm{rank}(\mathbf{P})\}$) largest eigenvalues of $\hat{\mathbf{R}}$, $\hat{\lambda}_1\geq\hat{\lambda}_2\geq\cdots\geq\hat{\lambda}_{\bar{K}}$, respectively, and

$$\hat{\tilde{\mathbf{\Lambda}}}=\mathrm{diag}\left\{\frac{(\hat{\lambda}_1-\hat{\sigma}^2)^2}{\hat{\lambda}_1},\frac{(\hat{\lambda}_2-\hat{\sigma}^2)^2}{\hat{\lambda}_2},\cdots,\frac{(\hat{\lambda}_{\bar{K}}-\hat{\sigma}^2)^2}{\hat{\lambda}_{\bar{K}}}\right\},\tag{18}$$

with

$$\hat{\sigma}^2=\frac{1}{M-\bar{K}}\left\{\mathrm{tr}\{\hat{\mathbf{R}}\}-\sum_{l=1}^{\bar{K}}\hat{\lambda}_l\right\}.\tag{19}$$

Let

$$[\tilde{\mathbf{s}}_1\ \cdots\ \tilde{\mathbf{s}}_{\bar{K}}]\triangleq\begin{bmatrix}\tilde{s}_{1,1}&\cdots&\tilde{s}_{1,\bar{K}}\\ \vdots&&\vdots\\ \tilde{s}_{M,1}&\cdots&\tilde{s}_{M,\bar{K}}\end{bmatrix}\triangleq\hat{\mathbf{S}}\hat{\tilde{\mathbf{\Lambda}}}^{\frac{1}{2}},\tag{20}$$

and

$$\mathbf{B}^H\tilde{\mathbf{s}}_k=\begin{bmatrix}\tilde{s}_{K+1,k}&\cdots&\tilde{s}_{1,k}\\ \vdots&&\vdots\\ \tilde{s}_{M,k}&\cdots&\tilde{s}_{M-K,k}\end{bmatrix}\mathbf{b}\triangleq\tilde{\mathbf{S}}_k\mathbf{b}.\tag{21}$$

MODE first obtains the initial estimate $\hat{\mathbf{p}}_0$ of $\mathbf{p}$ by

$$\hat{\mathbf{p}}_0=\arg\min_{\mathbf{p}}\left\|\begin{bmatrix}\mathrm{Re}\{\mathbf{H}_0\mathbf{W}\}\\ \mathrm{Im}\{\mathbf{H}_0\mathbf{W}\}\end{bmatrix}\mathbf{p}\right\|^2,\tag{22}$$

where

$$\mathbf{H}_0=\begin{bmatrix}\tilde{\mathbf{S}}_1\\ \vdots\\ \tilde{\mathbf{S}}_{\bar{K}}\end{bmatrix}\in\mathcal{C}^{\bar{K}(M-K)\times(K+1)},\tag{23}$$

and then obtains the refined estimate $\hat{\mathbf{p}}$ by

$$\hat{\mathbf{p}}=\arg\min_{\mathbf{p}}\left\|\begin{bmatrix}\mathrm{Re}\{\mathbf{H}\mathbf{W}\}\\ \mathrm{Im}\{\mathbf{H}\mathbf{W}\}\end{bmatrix}\mathbf{p}\right\|^2,\tag{24}$$

where

$$\mathbf{H}=\begin{bmatrix}\tilde{\mathbf{G}}^{-1}\tilde{\mathbf{S}}_1\\ \vdots\\ \tilde{\mathbf{G}}^{-1}\tilde{\mathbf{S}}_{\bar{K}}\end{bmatrix}\in\mathcal{C}^{\bar{K}(M-K)\times(K+1)},\tag{25}$$

and $\tilde{\mathbf{G}}$ denotes the Cholesky factor of matrix $(\tilde{\mathbf{B}}^H\tilde{\mathbf{B}})$ with $\tilde{\mathbf{B}}$ being formed with $\tilde{\mathbf{b}}=\mathbf{W}\hat{\mathbf{p}}_0$ and $\|\cdot\|$ denotes the Euclidean norm. The steps required by the MODE algorithm are:

Step 1: Compute $\hat{\mathbf{R}}=\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{y}(n)\mathbf{y}^H(n)$. This step requires $O\left[\frac{1}{2}M(M+1)N\right]$ flops.

Step 2: Compute the $\bar{K}$ dominant eigenvalues and the corresponding eigenvectors of $\hat{\mathbf{R}}$. This step requires about $O\left[2\bar{K}M^2\right]$ flops.

Step 3: Compute $\hat{\mathbf{p}}_0$. For simplicity, we assume that $\mathbf{u}=[\ 1\ \ 0\ \ \cdots\ \ 0\ ]^T$. Let

$$\mathbf{\Omega}_0=\begin{bmatrix}\mathrm{Re}\{\mathbf{H}_0\mathbf{W}\}\\ \mathrm{Im}\{\mathbf{H}_0\mathbf{W}\}\end{bmatrix}\triangleq[\ \mathbf{\Omega}_{01}\ \vdots\ \mathbf{\Omega}_{02}\ ],\tag{26}$$

where $\mathbf{\Omega}_{01}\in\mathcal{C}^{2\bar{K}(M-K)\times 1}$ and $\mathbf{\Omega}_{02}\in\mathcal{C}^{2\bar{K}(M-K)\times K}$. Let $\mathbf{\Omega}_{02}$ be QR decomposed as

$$\mathbf{\Omega}_{02}=[\ \mathbf{Q}_{01}\ \ \mathbf{Q}_{02}\ ]\begin{bmatrix}\mathbf{R}_0\\ \mathbf{0}\end{bmatrix},\tag{27}$$

where $\mathbf{Q}_{01}\in\mathcal{C}^{2\bar{K}(M-K)\times K}$, and $[\mathbf{Q}_{01}\ \ \mathbf{Q}_{02}]$ and $\mathbf{R}_0\in\mathcal{C}^{K\times K}$, respectively, are the orthogonal and upper triangular matrices. Then we have

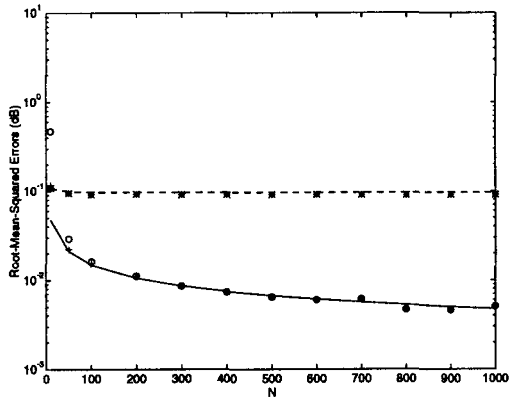
$$\hat{\mathbf{p}}_0=\begin{bmatrix}1\\ \hat{\mathbf{q}}_0\end{bmatrix},\tag{28}$$

with

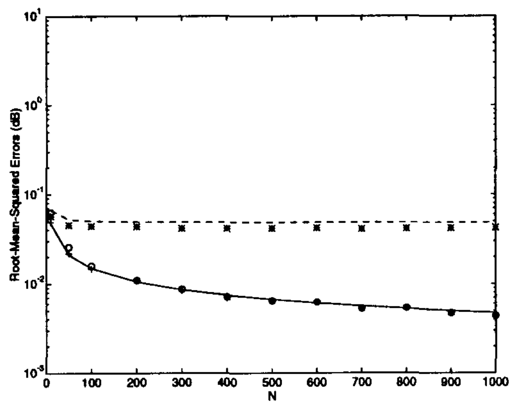
$$\hat{\mathbf{q}}_0=\mathbf{R}_0^{-1}\mathbf{Q}_{01}^H\mathbf{\Omega}_{01}.\tag{29}$$

Since $O\left[2\bar{K}(M-K)(K+1)^2\right]$ flops are required to compute $\mathbf{\Omega}_0=\mathbf{H}_0\mathbf{W}$, $O\left[\frac{2}{3}\bar{K}(M-k)K^2-\frac{1}{3}K^3\right]$ flops are required to obtain $\mathbf{Q}_{01}$ and $\mathbf{R}_0$, $O\left[\bar{K}(M-K)K\right]$ flops are required to compute $\mathbf{Q}_{01}^H\mathbf{\Omega}_{01}$, and $O\left[\frac{1}{2}K^2\right]$ flops are required to obtain $\hat{\mathbf{p}}_0$ by back substitution. This step requires about $O\left\{\bar{K}(M-K)\left[2(K+1)^2+\frac{2}{3}K^2+K\right]\right\}$ flops.

Step 4: Compute $\hat{\mathbf{p}}$. From Steps 1-3 of IQML and Step 3 of MODE, we note that $O\left[\frac{1}{2}(K+1)(K+2)\right]$ flops are required to compute $\tilde{\mathbf{C}}=\tilde{\mathbf{B}}^H\tilde{\mathbf{B}}$, $O\left[(M-K)(\frac{1}{2}K^2+\frac{3}{2}K)\right]$ flops are required to compute the Cholesky factor $\tilde{\mathbf{G}}$ of $\tilde{\mathbf{C}}$, $O\left[2\bar{K}(K+1)^2(M-K)\right]$ flops are required to compute $\mathbf{H}$, $O\left[2\bar{K}(M-K)(K+1)^2\right]$ flops are required to compute $\mathbf{\Omega}=\mathbf{H}\mathbf{W}$, and $O\left\{\bar{K}(M-K)\left[\frac{3}{2}K^2+(K+1)^2+K\right]\right\}$ flops are required to compute $\hat{\mathbf{p}}$ by minimizing $\|\mathbf{\Omega}\mathbf{p}\|^2$. Thus, this step requires about $O\left\{\bar{K}(M-K)\left[5(K+1)^2+\frac{3}{2}(K^2+K+1)\right]\right\}$ flops.



(a) For $\omega_1 = 2\pi(0.02)$.



(b) For $\omega_2 = 2\pi(0.10)$.

Figure 1. *RMSEs of estimates obtained by MODE ("o"), I-MODE ("+"), and IQML ("*") for coherent signals as a function of N when $M = 10$, $K = 2$, $\sigma^2 = 1$, and $\mathbf{P} = \mathbf{E}$ where all elements in $\mathbf{E}$ are 1's. The solid lines are for the CRB and the dashed lines are for the theoretical RMSE of IQML.*

For $N \gg M \gg K$, MODE requires about $O\left[\frac{1}{2}(M+1)MN\right]$ flops. Therefore, for $N \gg M \gg K$, the ratio γ between the numbers of the flops required by IQML and MODE is about

$$\gamma = \frac{(K+1)(3K+4)(M-K)L}{M(M+1)}. \qquad (30)$$

6. NUMERICAL EXAMPLE

Figure 1 shows the comparison of the CRBs and the theoretical RMSEs (Root-Mean-Squared Errors) of IQML as a function of N together with the corresponding RMSEs of the frequency estimates obtained by using MODE and IQML for $\omega_k = 2\pi f_k$, $k = 1, 2, \cdots, K$, via 100 Monte-Carlo simulations. We note that the RMSEs obtained with IQML and MODE, respectively, approach the theoretical RMSE of IQML and the CRB as N increases. The difference between the theoretical RMSE of IQML and the CRB becomes more significant as N increases since the IQML estimates are always biased. We also note that the performance of MODE is worse than that of IQML for $N < 20$. For this case, we can iterate Step 4 of MODE a few times to improve its performance. We refer to this approach as the *iterative MODE* or *I-MODE*. The RMSEs of I-MODE estimates in Figure 1 are obtained by iterating Step 3 of MODE 5 times. We note that the performance of I-MODE is better than that of IQML and MODE for small N. For large N, the performance of I-MODE is about the same as that of MODE and hence the iterations are not necessary.

For this example, IQML requires about 10 iterations on average. For $N = 10$, 100, and 1000, the number of flops required by IQML is, respectively, about 5, 18, and 23 times of that required by MODE and about 2, 12, and 21 times of that by I-MODE. These results approximately agree with the ratio given in Equation (30).

When σ^2 in the previous example is increased to 10 and all other parameters remain the same, the ratios between the number of flops required by IQML and those by MODE become 20, 60, and 80 for $N = 10$, 100 and 1000, respectively. This is because the average number of iterations required by IQML is increased to about 35 for this case.

7. CONCLUSIONS

We have presented a comparative study of using the IQML and MODE algorithms for DOA estimation with a ULA. The consistent condition and the theoretical performance of IQML have been presented. The computational complexities of both algorithms have also been compared. We have shown that the frequency estimates obtained via MODE are asymptotically statistically efficient, while those obtained via IQML are almost always inconsistent and hence inefficient. We have also shown that the amount of computations required by IQML is usually much larger than that required by MODE, especially for low SNR and large number of snapshots.

REFERENCES

[1] R. Kumaresan, L. L. Scharf, and A. K. Shaw, "An algorithm for pole-zero modeling and spectral analysis," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 34, no. 3, pp. 637-640, June 1986.

[2] Y. Bresler and A. Macovski, "Exact maximum likelihood parameter estimation of superimposed exponential signals in noise," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 34, no. 5, pp. 1081-1088, October 1986.

[3] P. Stoica and K. C. Sharman, "Maximum likelihood methods for direction-of-arrival estimation," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, no. 7, pp. 1132-1143, July 1990.

[4] Y. Hua, "The most efficient implementation of IQML algorithm," *IEEE Transactions on Signal Processing*, vol. 42, no. 8, pp. 2203-2204, August 1994.

[5] P. Stoica, J. Li, and T. Söderstrom, "On the inconsistency of IQML," to appear in *Signal Processing*.