PARTICLE VELOCITY AND SIZE ESTIMATION FROM TWO CHANNEL LASER ANEMOMETRY MEASUREMENTS

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ABSTRACT

A mathematical treatment of particle size and velocity estimation from two channel laser anemometry measurements is considered. Cramér-Rao bounds for the general case are derived, and the corresponding maximum likelihood estimator is analyzed through computer simulations. Low complexity correlation based estimators are derived and their performance is characterized. The results predicted by theory are illustrated by some numerical examples.

1. INTRODUCTION

Estimation of frequency and phase of a sinusoidal signal in additive noise is a classical problem of estimation theory in digital signal processing, which has a large number of applications. In some applications a frequency source is sampled through several sensors, resulting in output signals with the same frequency but with different phases, or equivalently, with different delays. One such application is phase doppler anemometry (PDA), on which the analysis in this paper is concentrated. PDA is used to simultaneously measure velocity and size of small spherical particles or dropplets, and is, as far as is known to the authors, the only technique capable of simultaneous measurement of velocity and size of individual particles. This fact makes it possible to correlate size and velocity in, for example, sprays. PDA is applied in a number of different fields: for spray measurements (medical, spray painting), to measure combustion in fuel injection engines and generators in power plants, for design of ink-jet printers, etc.

The basic principle of PDA is to focus one or two laser beams into a small volume, the measurement volume. When a particle or dropplet passes through the measurement volume, light is scattered in different directions. The Doppler shift of the light is proportional to the velocity of the particle, and is the same in all directions, while the phase differences between light scattered in different directions are related to the particle size. The signal processing problem in this application is thus to estimate frequency and phase of signals from different sensors, all with the same frequency, but with different phases. The SNR may also be different for the sensors, since the intensity of light scattered in different direction may vary many magnitudes of order.

The noise in the measurements mainly originates from two sources: photon noise and background noise due to electronic noise in amplifiers etc., and background illumination. While photon noise cannot be characterized as additive noise, see [1], the background noise usually is dominating in the most difficult measurement situations. The signals may therefore be modeled as sinusoidal signals in additive, white (Gaussian) noise.

In this paper, some methods for frequency and phase estimation are analyzed. The scope is restricted to two sensor systems, but the analysis may be extended to several sensors.

The measurements from the two sensors are processed by quadrature mixing and sampling, and are given by

$$\begin{aligned} z_1(k) &= s_1(k) + v_1(k), \qquad k = 0, \ldots, N-1 \\ z_2(k) &= s_2(k) + v_2(k) \end{aligned} \tag{1}$$

where $(s_1(k), s_2(k))$ are the signals of interest. The zero mean noise sequences $(v_1(k), v_2(k))$ are mutually uncorrelated, and fulfill

$$\begin{aligned} E[v_1(k)v_1^*(\ell)] &= 2\sigma_1^2\delta_{k,\ell}, \quad E[v_1(k)v_1(\ell)]=0 \quad \forall k,\ell \\ E[v_2(k)v_2^*(\ell)] &= 2\sigma_2^2\delta_{k,\ell}, \quad E[v_2(k)v_2(\ell)]=0 \quad \forall k,\ell \end{aligned} \tag{2}$$

where $*$ denotes complex conjugate, and $\delta_{k,l}$ is the Kronecker delta. The noise powers σ_1 and σ_2 may, or may not, be equal. For calculation of Cramér-Rao bounds (CRB) and for derivation of the maximum likelihood estimator (MLE), the noise will further be assumed Gaussian. In PDA the signals from the two channels have the same frequency, but different phases, that is

$$s_p(k) = \alpha_p e^{i(\omega k + \phi_p)}, \qquad p = 1, 2 \tag{3}$$

where (α_1, α_2) are the real-valued amplitudes, $\alpha_p > 0, p = 1, 2$, (ϕ_1, ϕ_2) the initial phases, $\phi_p \in [0, 2\pi], p = 1, 2$, and ω is the normalized radian frequency, $\omega \in (-\pi, \pi)$.

The frequency ω contains the information about the velocity, the relative phase $\psi = \phi_1 - \phi_2$ contains the information about the size of the scattering particle. Thus, the problem to be solved is estimation of ω and ψ from data (1). Although, a model (1)-(3) describing the measurements has been introduced, the considered estimation methods, except the MLE, do not explicitly rely on this model. However, in

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order to analyze their statistical properties (the error variance), a model of data is needed. In (1)-(3), the parameters $(\omega, \phi_1, \psi, \alpha_1, \sigma_1, \alpha_2, \sigma_2)$ are all assumed unknown.

2. CRAMÉR-RAO BOUND

In [2], the CRB is derived for two channel measurements, under the assumption that the SNR on the two channels are the same. This derivation is extended here to the case when the two channels have different SNR. The vector of unknown parameters is $\theta = (\omega, \phi_1, \psi, \alpha_1, \sigma_1, \alpha_2, \sigma_2)$. The elements of the Fisher information matrix are given by (see [3])

$$\mathbf{I}(\theta)_{i,j} = \frac{1}{\sigma_1^2}\sum_{k=0}^{N-1}\left(\frac{\partial\mu_1(k)}{\partial\theta_i}\frac{\partial\mu_1(k)}{\partial\theta_j} + \frac{\partial\nu_1(k)}{\partial\theta_i}\frac{\partial\nu_1(k)}{\partial\theta_j}\right) + \frac{N}{2\sigma_1^4}\frac{\partial\sigma_1^2}{\partial\theta_i}\frac{\partial\sigma_1^2}{\partial\theta_j} + \frac{1}{\sigma_2^2}\sum_{k=0}^{N-1}\left(\frac{\partial\mu_2(k)}{\partial\theta_i}\frac{\partial\mu_2(k)}{\partial\theta_j} + \frac{\partial\nu_2(k)}{\partial\theta_i}\frac{\partial\nu_2(k)}{\partial\theta_j}\right) + \frac{N}{2\sigma_2^4}\frac{\partial\sigma_2^2}{\partial\theta_i}\frac{\partial\sigma_2^2}{\partial\theta_j} \tag{4}$$

where

$$\begin{aligned} \mu_1(k) &= \alpha_1\cos(\omega k + \phi_1) \qquad k = 0, \ldots, N-1 \\ \nu_1(k) &= \alpha_1\sin(\omega k + \phi_1) \\ \mu_2(k) &= \alpha_2\cos(\omega k + \phi_1 - \psi) \\ \nu_2(k) &= \alpha_2\sin(\omega k + \phi_1 - \psi). \end{aligned} \tag{5}$$

Accordingly, the Fisher information matrix is

$$\begin{aligned} \mathbf{I}(\theta) &= \sum_{k=0}^{N-1}\begin{bmatrix} \mathbf{A}(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{B}(k) \end{bmatrix} \\ \mathbf{A}(k) &= \frac{\alpha_1^2}{\sigma_1^2}\begin{bmatrix} k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha_2^2}{\sigma_2^2}\begin{bmatrix} k^2 & k & k \\ k & 1 & 1 \\ k & 1 & 1 \end{bmatrix} \\ \mathbf{B}(k) &= \mathrm{diag}\left(1, \frac{1}{2\sigma_1^4}, 1, \frac{1}{2\sigma_2^4}\right). \end{aligned}$$

The resulting Fisher matrix is

$$\begin{aligned} \mathbf{I}(\theta) &= \sum_{k=0}^{N-1}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \\ \mathbf{A} &= \frac{\alpha_1^2}{\sigma_1^2}\begin{bmatrix} Q & P & 0 \\ P & N & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha_2^2}{\sigma_2^2}\begin{bmatrix} Q & P & P \\ P & N & N \\ P & N & N \end{bmatrix} \\ \mathbf{B} &= \mathrm{diag}\left(N, \frac{N}{2\sigma_1^4}, N, \frac{N}{2\sigma_2^4}\right) \end{aligned}$$

where $P = \sum_{k=0}^{N-1} k = N(N-1)/2$ and $Q = \sum_{k=0}^{N-1} k^2 = N(N-1)(2N-1)/6$. By inverting the Fisher matrix, the CRB is given by

$$\begin{aligned} \mathrm{CRB}[\hat{\omega}] &= \frac{6}{(\mathrm{SNR}_1 + \mathrm{SNR}_2)}\frac{1}{N(N^2-1)} \\ \mathrm{CRB}[\hat{\phi}_1] &= \frac{(4\mathrm{SNR}_1 + \mathrm{SNR}_2)N + (\mathrm{SNR}_2 - 2\mathrm{SNR}_1)}{2\mathrm{SNR}_1(\mathrm{SNR}_1 + \mathrm{SNR}_2)N(N-1)} \\ \mathrm{CRB}[\hat{\psi}] &= \frac{\mathrm{SNR}_1 + \mathrm{SNR}_2}{\mathrm{SNR}_1\mathrm{SNR}_2}\frac{1}{2N} \\ \mathrm{CRB}[\hat{\alpha}_p] &= \frac{\sigma_p^2}{2N} \qquad p = 1, 2 \\ \mathrm{CRB}[\hat{\sigma}_p] &= \frac{\sigma_p^4}{N} \qquad p = 1, 2 \end{aligned} \tag{6}$$

where $\mathrm{SNR}_p = \alpha_p^2/2\sigma_p^2$, $p = 1, 2$.

3. MAXIMUM LIKELIHOOD ESTIMATOR

The MLE was derived in [2]. The frequency is estimated by *minimizing* the following expression with respect to ω

$$[P_1 - |A_1(\omega)|^2][P_2 - |A_2(\omega)|^2] \tag{7}$$

or, by *maximizing* the following (positive) expression

$$P_1P_2 - [P_1 - |A_1(\omega)|^2][P_2 - |A_2(\omega)|^2] \tag{8}$$

where P_p and $A_p(\omega)$ are total power and Fourier transform of the two signals, respectively. Explicitly,

$$\begin{aligned} P_p &= \frac{1}{N}\sum_{k=0}^{N-1}|z_p(k)|^2 \qquad p = 1, 2 \\ A_p(\omega) &= \frac{1}{N}\sum_{k=0}^{N-1} z_p(k)\exp(-i\omega k). \end{aligned} \tag{9}$$

Once the MLE $\hat{\omega}$ has been found, the other parameters can be found from

$$\begin{aligned} \hat{\psi} &= \angle[A_1(\hat{\omega})] - \angle[A_2(\hat{\omega})] \\ \hat{\alpha}_p &= |A_p(\hat{\omega})|, \qquad p = 1, 2 \\ \hat{\sigma}_p &= P_p - |A_p(\hat{\omega})|^2. \end{aligned} \tag{10}$$

For the single channel case, the MLE can be implemented using FFT with a moderate zero-padding (2 or 4 times) followed by parabolic or Gaussian interpolation, with Gaussian giving less bias, [4]. For the two channel case considered here, interpolation directly on the cross-spectrum ((7) or (8)) does not seem to work well at high SNR. Two times zero-padding followed by parabolic interpolation gives sufficient accuracy up to 50 dB for the single channel case. To obtain the same accuracy for the two channel case with parabolic interpolation, 128 times zero-padding is needed. The reason is that the peak of the cross-spectrum (8) is not well-approximated by a parabola. If $|A_1(\omega)|^2$ and $|A_2(\omega)|^2$ are approximated by parabolas near the peak, (8) is a fourth order polynomial. It was found, however, that interpolation with a fourth order polynomial directly on (8) did not work well, nor did spline interpolation.

Instead the following strategy is employed. The minimum bin of (7) is found using FFT with four times zero-padding. A parabola is fitted to *each* spectrum $|A_1(\omega)|^2$ and $|A_2(\omega)|^2$ using the minimum bin of the cross-spectrum and the two surrounding bins, giving $p_1(\omega)$ respectively $p_2(\omega)$. The estimate of ω is then found as the mimimum of the fourth order polynomial $(P_1 - p_1(\omega))(P_2 - p_2(\omega))$.

The above strategy requires finding the roots of a third order polynomial (that is, the derivative of the polynomial $(P_1 - p_1(\omega))(P_2 - p_2(\omega))$), which can be done analytically. The analytic solution is, however, rather complicated. Furthermore, since the method is based on parabolic interpolation, the solution is not totally unbiased. Therefore, further investigations are needed to make the MLE feasible for real-time applications.

In order to find the phase (difference) from (10) the Fourier transform should be found at the MLE of the frequency $\hat{\omega}$. This can be done by directly evaluating the DFT at this frequency, but this is computationally intensive (ca. $4N$ (real) multiplications). Instead the DFT calculated by FFT is linearly interpolated, and the phases found from the interpolated spectrum. It is found that this gives high enough accuracy for the phase difference (see the results below).

4. PHASE ESTIMATOR

When estimation of phase is sufficient, the following correlation based estimator may be used,

$$\begin{aligned} \hat{\psi} &= \angle[\hat{R}_\psi] \\ \hat{R}_\psi &= \frac{1}{N}\sum_{k=0}^{N-1} z_1(k) z_2^*(k). \end{aligned} \quad (11)$$

The estimator is motivated by the fact that for noiseless measurements $R_\psi = \sum_{k=0}^{N-1} s_1(k)s_2^*(k)/N = \alpha_1\alpha_2 e^{i\psi}$. Thus, it directly follows that $\hat{\psi} \to \psi$ for increasing SNR, that is $\sigma_1, \sigma_2 \to 0$. For white Gaussian measurement noise one can show that $E[\hat{R}_\psi] = R_\psi$ meaning that the estimator (11) is consistent, that is $\hat{\psi} \to \psi$ when $N \to \infty$. The asymptotic variance of $\hat{\psi}$ can be found using the technique in [5]. The result is,

$$\mathrm{var}[\hat{\psi}] = \frac{\mathrm{SNR}_1 + \mathrm{SNR}_2 + 1}{2N\mathrm{SNR}_1\mathrm{SNR}_2} = \mathrm{CRB}[\hat{\psi}]\left(1 + \frac{1}{\mathrm{SNR}_1 + \mathrm{SNR}_2}\right) \quad (12)$$

where (6) was used in the last equality. From (12) it follows that as SNR $\to \infty$ an unity ratio of the error variance to the CRB is reached, but (11) is not asymptotically ($N \to \infty$) statistically efficient. For $\mathrm{SNR}_p = 0$dB, $p = 1, 2$, the loss in efficiency, that is the quotient of the error variance divided by the CRB, is given by 3/2 (1.8dB).

5. FREQUENCY ESTIMATION

In situations where also the frequency is of interest it may be estimated using correlation based methods. Here, the study is restricted to methods that utilize first lag sample correlations. In contrary to the case of phase estimation, frequency estimators based on a first lag sample correlation exhibit an error variance that significantly may exceed the CRB, especially when the quotient of the number of data N divided by the signal to noise ratio is large, [7]. On the other hand, when the quotient is small they may have excellent performance.

5.1. Single channel estimation

Based on a single channel measurement $z_p(k)$, $p = 1, 2$, the frequency may be estimated using a weighted linear predictor frequency estimator, [6]

$$\begin{aligned} \hat{\omega}_p &= \angle[\hat{R}_{\omega,p}], \quad p = 1, 2 \\ \hat{R}_{\omega,p} &= \sum_{k=1}^{N-1} W_k\, z_p(k) z_p(k-1)^* \end{aligned} \quad (13)$$

with W_k a weighting function such that $\sum_{k=1}^{N-1} W_k = 1$. With $W_k = 1/(N-1)$ the method is refered as the Linear Predictor (LP), and with $W_k = 6k(N-k)/(N(N^2-1))$ as the Weighted Linear Predictor (WLP), [6]. These two estimators result in estimates with error variance

$$\mathrm{var}[\hat{\omega}_p]_{LP} = \frac{1}{\mathrm{SNR}_p(N-1)}\left(\frac{1}{N-1} + \frac{1}{2\mathrm{SNR}_p}\right) \quad (14)$$

and

$$\mathrm{var}[\hat{\omega}_p]_{WLP} = \frac{6}{\mathrm{SNR}_p\, N(N^2-1)}\left(1 + \frac{N^2+1}{10\,\mathrm{SNR}_p}\right). \quad (15)$$

From (15) it follows that as SNR $\to \infty$ an unity ratio of the error variance to the CRB is reached, but WLP is not asymptotically ($N \to \infty$) statistically efficient. For LP the quotient of the error variance divided by the CRB is proportional to $N/6$ at high SNR, and thus the loss in performance may be substantial, [6]. For low SNR or large N, on the other hand, LP slightly outperforms WLP, and the variance of the former tends (as $N \to \infty$) to the lowest variance of any consistent frequency estimator based on first lag sample correlations, [7]. For finite N and SNR, other data windows may exist that produce estimates with lower error variance than LP/WLP, [5].

Based on (13), the frequency estimator based on measurements from both channels that minimizes the error variance is

$$\hat{\omega} = \sum_{p=1}^{2} \frac{\hat{\omega}_p}{\mathrm{var}[\hat{\omega}_p]} \Big/ \sum_{p=1}^{2} \frac{1}{\mathrm{var}[\hat{\omega}_p]}. \quad (16)$$

The estimator (16), however, requires knowledge of the SNR in the different channels, that is not apriori known. Assuming equal channel SNR, (16) reduces to $\hat{\omega} = 0.5(\hat{\omega}_1 + \hat{\omega}_2)$.

5.2. Dual channel estimation

Based on an estimate of ψ, let $z(k) = z_1(k) + e^{i\hat{\psi}} z_2(k)$, $s(k) = s_1(k) + e^{i\hat{\psi}} s_2(k)$, and $v(k) = v_1(k) + e^{i\hat{\psi}} v_2(k)$. Then $z(k)$ equals a noisy cisoid with complex-valued amplitude $\alpha = \alpha_1 e^{i\phi_1} + \alpha_2 e^{i(\phi_1 + \delta\psi)}$ where $\delta\psi = \hat{\psi} - \psi$, and the noise has variance $2\sigma^2$ where $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Using $z(k)$ as input data to LP/WLP in (13) results in an estimate with error variance (14)/(15), with SNR_p there replaced by SNR $= (\alpha_1^2 + 2\alpha_1\alpha_2\cos(\delta\psi) + \alpha^2) \,/\, 2(\sigma_1^2 + \sigma_2^2)$. Since both sensors are located in the same measurement volume, a reasonable assumption in PDA is that $\sigma_1^2 \approx \sigma_2^2$, and thus the effective SNR is

$$\mathrm{SNR} = \frac{\mathrm{SNR}_1 + 2\sqrt{\mathrm{SNR}_1\mathrm{SNR}_2}\cos\delta\psi + \mathrm{SNR}_2}{2}. \quad (17)$$

The total SNR is here dependent on the cosine of the relative phase error, that is a small quantity proportional to $1/\sqrt{N}$.

6. COMPUTER SIMULATIONS

In Figure 1, the performance of the correlation based phase estimator (11) and the MLE (implemented as described in Section 3) are displayed for $N = 32$ and $\psi = \pi/3$, where ϕ_1 is generated as an uniform random number on $(0, 2\pi)$. The estimated variances are based on 1000 independent simulation runs. The CRB and theoretical error variance are

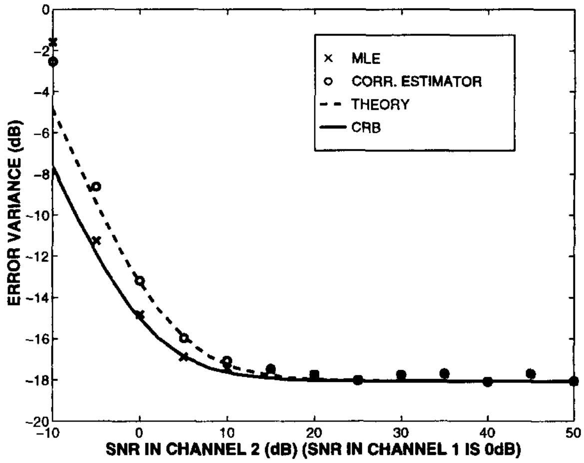

Figure 1: The empirical *phase error variance* as function of SNR in channel 2. The performance of MLE and (11) are shown. The CRB and the theoretical variance (12) are also displayed.

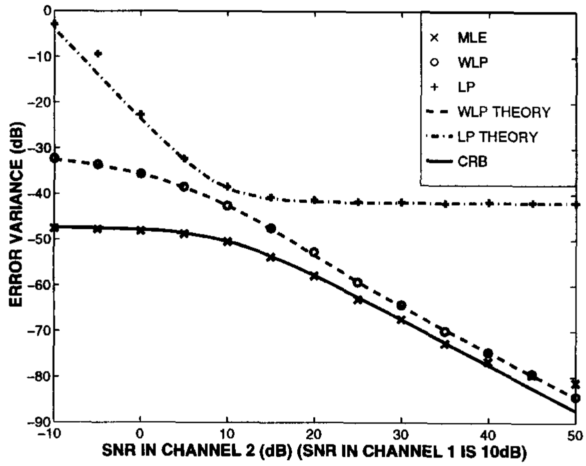

Figure 2: The empirical *frequency error variance* as function of SNR in channel 2 for MLE, the two channel averaged LP estimate, and the WLP estimate based on phase corrected averaged data. The CRB and the theoretical variances for LP/WLP are also displayed. For WLP, the theoretical error variance is calculated with $\delta\psi$ in (17) replaced with $(\text{CRB}(\hat{\psi}))^{-1/2}$.

also displayed, (6) and (12), respectively. One may note an excellent agreement between the empirical results and the results predicted from theory, for all SNR above -10dB in channel 2 where some outlier estimates result in a deviation between estimated and predicted error variance.

In Figure 2, the performance of the MLE, the LP estimator (13) with $\hat{\omega}$ given by $\hat{\omega} = 0.5(\hat{\omega}_1 + \hat{\omega}_2)$, and the WLP estimator (13) for $z(k) = z_1(k) + e^{i\hat{\psi}} z_2(k)$, is illustrated for $N = 32$ and ϕ_1, ϕ_2 uniformly distributed on $(0, 2\pi)$. The phase estimate used for phase corrections of $z_2(k)$ is given by (11). For this scenario, there is only a minor difference between LP and WLP, and thus the two channel WLP and the LP based on the phase corrected sum of data are excluded from the comparison. The statistics are based on 1000 independent simulation runs. One may again note an excellent agreement between predicted and actual results. The deviation of the MLE from the CRB for $\text{SNR}_2 > 40$dB is due to the interpolation. If 8 times zero-padding is used instead of 4 times, the MLE follows the CRB beyond 50dB. It can be noticed that the gain of using the MLE rather than WLP is limited for high SNR, although the computational complexity of the MLE is considerably higher. However, at low SNR the MLE should outperform the WLP.

7. CONCLUSIONS

A mathematical treatment of particle size and velocity estimation from two channel measurements has been considered. In terms of two channel phase and frequency estimation this problem is characterized by short data records ($N = 16 - 256$), a large variation of the SNR (typically in the range -10 to $+50$dB), as well in the SNR ratio between the different channels.

The study has been restricted to infinite precision arithmetics. Since the amplitude of the measured signal may vary of order 50dB within a short duration, a practical problem is that the gain of the pre-amplifiers have to be adjusted to the range of the A/D converters, alternatively processing of 1-bit (sign) sampled data. As shown in [2], 1-bit processing for phase estimation gives a large estimation error, while it performs reasonably for frequency estimation. A possible method is therefore to use 1-bit processing for frequency estimation combined with the estimator (11) for phase estimation applied on multibit data.

The CRB has been derived, the performance of the ML estimator has been studied, and some low complexity correlation based size and velocity estimators have been proposed. In particular, it has been shown that if only the size of the particle is of interest, a simple correlation based estimator provides accurate results. Velocity estimation from correlations are also possible based on a size estimate.

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