ON SOME PARAMETER ESTIMATION PROBLEMS IN ALPHA-STABLE PROCESSES

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ABSTRACT

Current algorithms for estimating the parameters of a symmetric alpha-stable ARMA process are either highly non-linear, or assume small MA orders ($q \leq 3$), or invoke the minimum-phase assumption. We use results from the statistics literature to show that the *normalized* correlation is well-defined; we show that the *normalized* cumulants are also well-behaved. We propose to use the correlation to estimate the spectrally-equivalent minimum-phase (SEMP) parameters, and then to use the cumulants to resolve the phase of the model. We also show that correlation-based techniques (such as ESPRIT) work well for estimating the parameters of harmonics observed in alpha-stable noise. Correlation-based algorithms are shown to work well despite the infinite variance of the alpha-stable process.

1. INTRODUCTION

In typical signal processing applications, additive noise can often be well modeled as the sum of a nominal stationary Gaussian component (thermal noise, etc) and high amplitude non-stationary, non-Gaussian (NG) components. Traditional models for the NG component include the Middleton model and its simplified versions such as the Gaussian-mixture model, the K-distribution, etc.

Over the last few years, lot of attention has been given to the stationary symmetric alpha stable (SαS) process. The characteristic function of the SαS random variable (rv) is given by [9], $E\{\exp(jvx)\} = \exp(-\sigma^\alpha|v|^\alpha)$, $0 < \alpha \leq 2$. Parameter α is called the index or characteristic exponent, and $\gamma = \sigma^\alpha$, $\sigma > 0$, is called the dispersion or scale parameter. Apart from the lack of a closed-form pdf ($\alpha \neq 1, \alpha \neq 2$), the S$\alpha$S rv possesses various interesting properties such as, tail probabilities of order $|x|^{-\alpha}$ and $E|x|^p = \infty$, $p > \alpha$, $\alpha < 2$, etc [7, 9]. Note that the natural sample estimates of $|x|^p$ are consistent only for $-1/2 < p < \alpha/2$. These and other features of the SαS pdf often appear to make parameter estimation problems intractable. It should also be pointed out that the limiting case $\alpha = 2$, the Gaussian, is in several ways, a 'discontinuity'. An overview of alpha-stable processes may be found in [7, 9].

We address two problems in this paper: the blind identification of a mixed-phase linear system excited by an iid SαS random process (rp), and estimation of the parameters of sinusoids observed in SαS noise. The former may be useful in modeling correlated impulsive noise.

2. COVARIATION OR CORRELATION ?

Since the SαS rv has 'infinite variance', the *covariation* defined by [9, p 94],

$$C_\alpha(x,y) := \gamma_y E\{X|Y|^{p-2}Y^*\}/E|Y|^p, 1 \leq p < \alpha \leq 2 \ ,$$

has often been used instead of the correlation, e.g., [1, 4, 7]. The covariation is well-defined only if X and Y are jointly alpha-stable, which means, in particular, that they both have the same α, with $1 < \alpha \leq 2$. Thus, one cannot define the covariation of a Cauchy rv with a Gaussian rv. Further both X and Y must be real, or both must be *isotropic complex*. The definition of the covariation also depicts an interesting Bussgang-type property of SαS rv's, namely that all fractional cross-moments are equivalent.

The *covariation coefficient* is the normalized covariation, and is defined as [7, 9]

$$\lambda_{xy} := E\{X|Y|^{p-2}Y^*\}/E|Y|^p \ , 1 \leq p < \alpha \leq 2 \ . \qquad (1)$$

For a SαS rp, $x(n)$, define

$$\lambda_{xx}(m) := \lambda_{x(n),x(n+m)} \ .$$

Note again that the covariation coefficient is defined only for $1 < \alpha \leq 2$. Suppose that $u(n)$ is an iid alpha-stable process; we see that $\lambda_{uu}(m) = \delta(m)$. It is tempting to use this as a test for the iid nature. Consider the natural sample estimate of $\lambda_{uu}(m)$:

$$\hat{\lambda}_{uu}(m) := \sum_{n=1}^{N} u(n)|u(n+m)|^{p-2}y^*(n+m) \ \Big/ \ \sum_{n=1}^{N}|u(n)|^p \ .$$

note that for $m \neq 0$, the variance of the sample estimate will be infinity because $E|u|^2 = \infty$. In [4] it is proposed (in the context of MA parameter estimation) that a value of $p < 1$ be used; specifically, in their simulations they use $p = 0.3$. But this does not solve our problem. From our extensive simulations, it seems that smaller values of p lead to estimates with greater variability. The *correlation coefficient* (or normalized correlation) which corresponds to $p = 2$ in (1) seems to be very stable. The natural estimate of the normalized correlation is

$$\hat{R}_{xx}(m) := \sum_{n=1}^{N} x(n)x^*(n+m) \ \Big/ \ \sum_{n=1}^{N}|x(n)|^2 \qquad (2)$$

We generated $N = 500$ samples of an iid SαS process, $u(n)$, with $\alpha = 1.5$; we also generated the AR(2) process, $y(n) = -0.75y(n-2) + u(n)$. We estimated the covariation (with $p = 0.3$) and the correlation sequences; estimates from $M = 50$ independent realizations are shown in Fig 1 for the two processes. In both cases, the correlation estimate has low variance; in contrast, the covariation estimate is not stable. Similar results were obtained for other values of p. The figures indicate that the covariation may not be useful for estimating model orders or model parameters.

The correlation estimator is useful *even* when $\alpha \leq 1$, in which case the covariation cannot be defined. The top

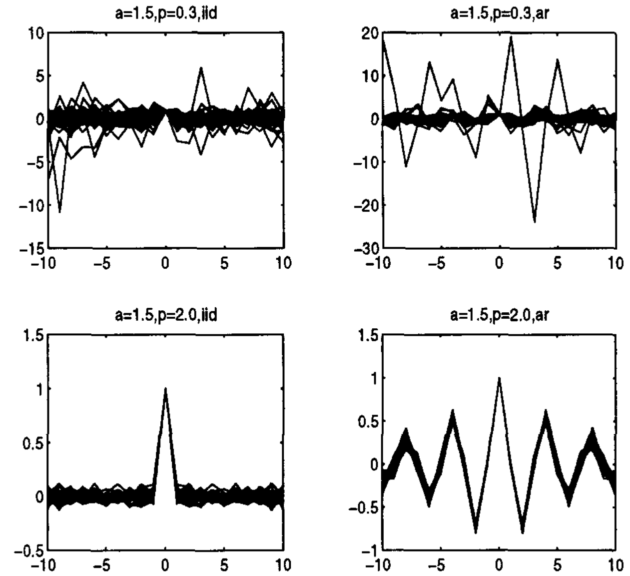

Figure 1. Estimates of the covariation with $p = 0.3$ (top panel) and correlation (bottom panel) of iid and AR SαS processes. Notice that the correlation estimates have low variance.

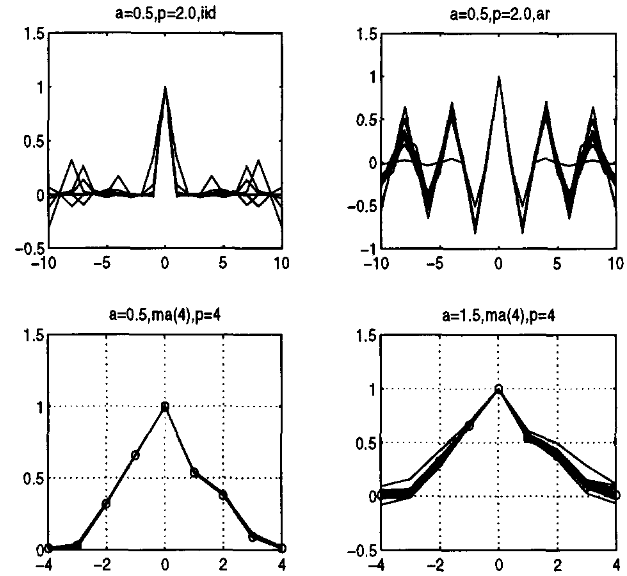

Figure 2. Top: Correlation estimate of an iid and an AR SαS process with $\alpha = 0.5$. Bottom: True and estimated 4-th order cumulant slice of an SαS MA(4) process, with $\alpha = 0.5$ and $\alpha = 1.5$.

two panels of Fig 2 show the correlation estimate for an iid SαS process and an AR(2) SαS process, with $\alpha = 0.5$ (all other parameters as in Fig 1). Again, note that the estimates are well behaved, although somewhat noisier than the estimates in Fig 1.

Since the correlation estimate appears to be well behaved, it is tempting to consider higher-order moments, cumulants and the corresponding spectra. We simulated a mixed-phase MA process, $x(n) = \sum_{k=0}^{q} b(k)u(n-k)$, $n = 1, ..., N$, where $u(n)$ is an iid SαS rp. We let $b = [1, 2, 3, 1, 1]$; sample estimates obtained from $N = 4000$ samples and 20 realizations are shown in the bottom panels of Fig 2, for $\alpha = 0.5$ and $\alpha = 1.5$, with $p = 4$. The circles show the normalized diagonal slice of the true fourth-order cumulant corresponding to the mixed-phase MA model; notice the excellent fit and the low variance of the estimates.

3. CONVERGENCE RATES [2,7]

Consider the linear process, $x(n) = \sum_{m=-\infty}^{\infty} h(m)u(n-m)$, where $u(n)$ is an iid SαS process, and impulse response (IR) $\{h(n)\}$ satisfies an alpha-summability condition. The limiting behavior of the sample estimate, $\hat{R}_{xx}(m)$ in (2), has been studied in [2], where it is shown that the estimate converges in probability to

$$R_{hh}(m) := \sum_{j=-\infty}^{\infty} h(j)h(j+m) \Bigg/ \sum_{j=-\infty}^{\infty} h^2(j) \ ,$$

which is the deterministic correlation of the IR. Further, it is shown in [2] that, for $0 < \alpha < 2$,

$$\left(\frac{n}{\log n}\right)^{1/\alpha} \left(\hat{R}_{xx}(m) - R_{hh}(m)\right) \longrightarrow A(m)V \ ,$$

where the convergence is in distribution; rv V can be expressed as the ratio U_1/U_2, where U_1 is a SαS rv with index α, independent of rv U_2 which is a positive alpha-stable rv with index $\alpha/2$. Both U_1 and U_2 have infinite variances, but their ratio V has finite variance. The non-random constant $A(m)$ is the *dispersion* given by

$$\left(\sum_j |R_{hh}(j+m) - R_{hh}(j-m) - 2R_{hh}(j)R_{hh}(m)|^{\alpha}\right)^{1/\alpha} .$$

Note that the rate of convergence is much faster than that for the Gaussian case.

The results of [2] were recently extended in [6], where they consider the Whittle-type estimator based on the *self-normalized periodogram*; they establish weak consistency of the estimator, and derive the same rates of convergence for the ARMA parameters, as were derived for $R_{xx}(m)$ in [2].

An intuitive explanation of the ease of estimating the SEMP parameters is as follows: the energy normalization suppresses all but the biggest impulses; thus the normalized output consists of (depending upon α, relatively) isolated impulse responses. In addition, the parameter estimation techniques are consistent in the presence of additive noise (Gaussian or non-Gaussian, white or colored) with finite variance. Thus, estimating the parameters of a linear SαS model is easier for $\alpha < 2$ than it is for $\alpha = 2$, the Gaussian case.

4. MA PARAMETER ESTIMATION

Based on the results of [2] and [6], we propose to use the correlation to estimate the spectrally-equivalent minimum-phase (SEMP) model, and then to resolve the true zeros by cumulant matching. Note that the 'SE' part of 'SEMP' is appropriate in view of the results in [6]. The unobserved input process, $u(n)$, is assumed to be iid SαS , and the observed signal is

$$y(n) = \sum_{k=0}^{q} b(k)u(n-k) \ , \quad n = 1, ..., N. \qquad (3)$$

Closed form expressions have been developed for $q \leq 3$ in [4]; the solution has a 0/0 problem for $q = 2$. A covariation-matching method was also proposed in [4], where it is noted that covariation-matching may lead to non-unique estimates because of the multi-modality of the cost function.

A non-parametric method, based on the so-called *alpha-spectrum* and the differential cepstrum, was derived in [4]; the algorithm involves computation of the Z-transform on two circles within the region of convergence (ROC). In addition, the algorithm involves truncation approximations, knowledge of the ROC, and knowledge of α; further, consistency has not been established. Finally, for the typical case of real signals and real channels, the method requires that the phase of the alpha-spectrum be continuous.

We note that the alpha-spectrum method and the covariation-based methods assume that the processes are SαS ; their performance, for non SαS processes, or in the presence of additive noise (e.g., sensor noise) is unknown. The correlation-based methods, on the other hand, are universally applicable.

Examples: We generated the process in (3), for various MA models, and SαS inputs with various values of α. Because of the inherent sign ambiguity, we assumed $b(0) = 1$. The sample correlation in (2) was used to fit a long AR model of order $K-q$, where K is the number of correlation lags used. The MA model parameters were then obtained via, $\sum_{k=0}^{q} b(k)a(n-k) = \delta(n),\ n = 0, ..., p$. There are, of course, much more sophisticated algorithms for estimating the MA parameters; the point here is not to stress a particular parameter estimation algorithm, but to demonstrate that correlation-based techniques work very well even when the input is SαS . Tables 1-3 show the mean and standard-deviation of the estimates, averaged over a set of $M = 100$ realizations. Results are shown for two values of α: $\alpha = 0.5$ and $\alpha = 1.5$ (other values of α yield similar results); N is the number of samples, and K is the number of lags. Note that the correlation-based method yields unbiased and low variance estimates of the minimum-phase (MP) equivalent model.

	b(1)	b(2)	b(3)
true	-4.4000	-1.6800	-0.3200
MP	0.1567	-0.0098	-0.0141
N=2000,M=100,K=12,$\alpha = 0.5$			
mean	0.1565	-0.0100	-0.0137
std	0.0012	0.0027	0.0078
N=2000,M=100,K=12,$\alpha = 1.5$			
mean	0.1533	-0.0112	-0.0153
std	0.0188	0.0161	0.0234

Table 1. Estimates for MA(3) model

	b(1)	b(2)	b(3)	b(4)
true/MP	-1.6000	1.9000	-1.0000	0.4000
N=2000,M=100,K=20,$\alpha = 0.5$				
mean	-1.5889	1.8496	-0.9990	0.4058
std	0.0669	0.1434	0.0890	0.0431
N=2000,M=100,K=20,$\alpha = 1.5$				
mean	-1.5844	1.8444	-0.9886	0.4003
std	0.0345	0.0467	0.0423	0.0245

Table 2. Estimates for MA(4) model

	b(1)	b(2)	b(3)	b(4)
true	-1.9600	2.8000	-1.6500	0.7500
MP	-1.6205	1.9227	-1.0156	0.3999
N=2000,M=100,K=30,$\alpha = 0.5$				
mean	-1.6109	1.9049	-1.0153	0.4038
std	0.1121	0.1948	0.1071	0.0426
N=2000,M=100,K=30,$\alpha = 1.5$				
mean	-1.6139	1.9011	-1.0060	0.3899
std	0.0365	0.0760	0.0574	0.0352

Table 3. Estimates for MA(4) model

The preceding examples show that standard correlation-based ("Gaussian") software packages may be used to estimate the equivalent minimum-phase model. How do we resolve the zeros? If $\alpha > 1$, one could try covariation matching; however, it is not clear that this will lead to unique estimates, and we have already noted the high variance of the covariation based estimators.

The cumulant estimates in the bottom panel of Fig 2 exhibit low variance; hence, cumulant-based techniques are indicated. Corresponding to the q zeros, there are (at most) 2^q spectrally equivalent mixed-phase models. The true zeros can be resolved via cumulant matching,

$$\min_k \sum_{\ell=0}^{q} \sum_{m=0}^{\ell} \sum_{n=0}^{m} |\hat{C}_{4y}(\ell, m, n) - C_{4h}(\ell, m, n|\theta_k)|^2 \qquad (4)$$

where $\hat{C}$ are the (normalized) cumulants estimated from the data, and $C(\cdot|\theta)$ are the cumulants corresponding to the model θ, and are given by,

$$C(\ell, m, n|\theta) = \sum_{s=0}^{q} b(s)b(s+\ell)b(s+m)b(s+n)\ . \qquad (5)$$

The technique can be extended to non-causal ARMA processes provided q is replaced by $L = 3p + q$ in (4), and the sum in (5) is over $s = -Q : Q$, $Q \gg p + q$. In this case, we have at most 2^{p+q} competing models. This is in the spirit of the cumulant matching algorithms of Tugnait, and Giannakis *et al* [5]; their proofs of uniqueness and consistency are applicable (after normalization). We quote from [2]: "Method of moment type estimates of the ARMA parameters will be weakly consistent regardless of α". Note that inherent all-pass factors cannot be resolved by this method, but can be handled by an extension given in [10].

A covariation-correlation based LS algorithm can also be developed. Consider:

$$R(\tau) := \sum_{n=0}^{q} b(n)b(n+\tau) \qquad C(\tau) := \sum_{n=0}^{q} f(n)b(n+\tau)$$

$$\sum_{k=0}^{q} f(k)R(k+\tau) \quad = \quad \sum_{n=0}^{q} b(n)C(n+\tau)$$

from which LS estimates of $\{b(n)\}_{n=1}^{q}$ and $\{f(n)\}_{n=0}^{q}$ can be obtained; the matrix will be full-rank if $F(z)$ and $B(z)$ do not have any common zeros. As an example, if $b(n)$ takes on only the values $0, \pm 1$, the matrix will not have full rank. If $C(\tau)$ are the covariations, then, $f(n) = |b(n)|^{\alpha-2}b^*(n)$. Choosing $f(n) = b^{\ell-1}(n)$, $\ell = 3, 4$ and the diagonal slice of the third- or fourth-order cumulants leads to the 'GM' algorithm [5].

5. AR PARAMETER ESTIMATION

We wish to demonstrate that conventional correlation-based techniques are consistent estimators of the AR parameters of a linear SαS process; further, we show that cumulant-based estimators may offer advantages when the SαS signal is corrupted by additive Gaussian noise.

Example. The observed signal was $y(t) = u(t)/A_I(z) + g(t)/A_n(z)$, where signal $u(t)$ was iid SαS with $\alpha = 1, \gamma = 1$; noise $g(t)$ was iid Gaussian with variance $\sigma_g^2 = 1$; the AR parameters were $A_I = [1, 0, 0.75]$ and $A_n = [1, -0.4, 0.6]$. We estimated the A_I parameters via the 'normal' equations based on *conventional* second- and fourth-order cumulants. Mean and standard deviations estimated from 100 trials, with $N = 1024$ samples, are shown in columns 2 and 3 of Table 4. Corresponding estimates with $\sigma_g^2 = 100$, $A_n = [1, 0, -0.75]$, $N = 4096$ (all other parameters same) are shown columns 4 and 5 of Table 4. The generalized SNR (GSNR), $10\log_{10}(\gamma_u/\sigma_g^2)$ is 0 dB for the first case, and -20 dB for the second case. *The correlation-based method* yields good estimates; the cumulant-based estimators are less biased at low GSNR.

	Noise-free		Noisy	
true	0.0000	0.7500	0.0000	0.7500
correlation-based				
mean	-0.0025	0.7454	-0.0014	0.7144
std	0.0136	0.0201	0.0129	0.0561
cumulant-based				
mean	-0.0018	0.7452	-0.0001	0.7373
std	0.0101	0.0365	0.0143	0.0599

Table 4. AR(2) model estimates.

6. HARMONICS IN SαS NOISE

In [1], covariations (with $p = 0.8$) were used to estimate the frequencies of real harmonics observed in iid SαS noise; they proposed the use of subspace methods based on the covariation.

Based on the results of [2, 6], it seems that standard correlation/periodogram based estimators should yield good results. We used the time-domain version of correlation-based ESPRIT; each realization was normalized to unit energy before the temporal correlation matrix was estimated. Since second-order statistics are used (hence, phase randomization is not important), multiple realizations can be created by segmentation.

Example. The observed signal was $y(n) = A(1)\exp(j2\pi f_1 n) + A(2)\exp(j2\pi f_2 n) + u(n)$, $n = 1, ..., N$, where $A(1)$, and $A(2)$, the amplitudes, were zero-mean unit variance Gaussian rv's (thus, amplitudes are fixed for a given realization, they vary from realization to realization), and $u(n)$ is iid SαS noise with unit dispersion. We set $f_1 = 0.1$, $f_2 = 0.2$, $N = 40$, and $M = 16$ realizations (thus, $40 \times 16 = 640$ samples for each estimate). In addition to standard ESPRIT, we also passed the observed signals through the zero memory non-linearity (ZMNL) described in [8], and then used ESPRIT on the clipped signal. Pre-processing the data by using a signed fractional power, such as in the covariation, is equivalent to using a particular ZMNL, but is not optimal in any sense. The top panel of Fig 3 shows the standard deviation (from 100 Monte Carlo runs) of the estimates as a function of α. The dashed line corresponds to using the ZMNL pre-processor. The ZMNL is particularly useful for $\alpha > 0.5$; is performance can be improved by using alpha-adaptive estimates of the clipping level. We repeated the simulation with random phases, instead of random amplitudes; results are shown in the bottom panel of Fig 4. Note that the GSNR is 0 dB for both cases.

7. CONCLUSIONS

One objective of this paper was to introduce the reader to some rather interesting results on the convergence rates of the normalized *correlation* sequence of an SαS ARMA process, and to demonstrate that correlation-based techniques yield unbiased and low-variance estimates of the SEMP ARMA model parameters. Further, we showed via simulations that the normalized higher-order moments and cumulants are also well behaved. Energy normalization and simple ZMNL pre-processing were also seen to be useful for estimating the parameters of a harmonic process observed in SαS noise. We saw that estimates of the covariation appear to show much more variability than the correlation; this, perhaps, explains why covariation-based methods appear to need relatively larger amounts of data.

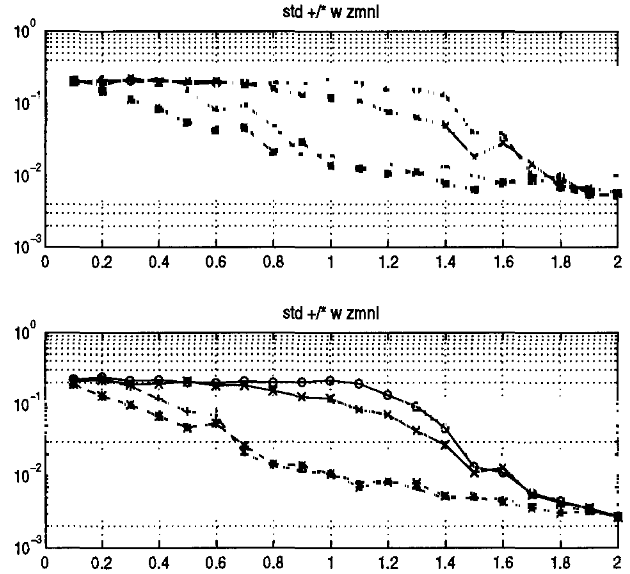

Figure 3. Correlation-based ESPRIT estimates with (dashed) and without (solid) ZMNL pre-processing. (a) Top: random amplitudes; (b) Bottom: random phases. Standard deviations are shown as a function of α.

Further extensions of the theory, and applications to the estimation of time-delays, frequency offsets, direction of arrival, and linear system parameters of/in SαS noise, with $0 < \alpha \le 2$ (not just $1 < \alpha \le 2$) may be found in [10], where Cramer-Rao bounds are also established.

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