SOME PROPERTIES AND ALGORITHMS FOR FOURTH ORDER SPECTRAL ANALYSIS OF COMPLEX SIGNALS

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ABSTRACT

In this paper, we give two algorithms for linear system blind identification based on the fourth order spectrum (or trispectrum). The first algorithm uses only N of the N^3 data of the fourth order spectrum. The second algorithm uses all the information contained in the fourth order spectra, but gives an optimal solution. This solution needs a previous phase unwrapping step; we give different solutions to unwrap the trispectrum phase. Finally, we establish the link between the well-known kurtosis maximization method and the optimal solution presented here; they are equivalent in first approximation. It means that we give an analytic solution to the blind identification problem which is nearly equivalent to the kurtosis maximization solution.
Keywords : HOS - Fourth Order Statistics - Blind Identification.

1. INTRODUCTION

Fourth order statistics of complex signals are among the possible tools for treating the problem of channel identification and equalization in digital communication [1]. Parametric methods (in the time domain) have been studied for several years [2],[1]; it seems that the fourth order methods in the frequency domain have not been studied in depth. We know the works of Mendel [1] Dalle Molle and Hinich [3], Pan and Nikias [4] and Le Roux *et al.* [5],[6]. Pierce [7],[8] and Shalvi and Weinstein [9] are among the few who treated the case of complex signals with applications in the field of radar signals and equalization respectively. Here, we extend a Fourier transform phase reconstruction algorithm that we developed in the case of third order spectra [10]. This multiresolution method does not raise phase unwrapping difficulties. If we intend to use optimal techniques [5][6], phase unwrapping is necessary : the trispectrum phase, being the sum of four spectrum phases (see eq. 3), takes values in the interval $[-4\pi, 4\pi]$; but the trispectrum phase is computed as the argument of the complex trispectrum then it is wrapped in the interval $[-\pi, \pi]$. Some algorithms, especially those involving divisions, require a phase unwrapping step. Here we give a phase unwrapping method that extends the work of Marron *et al.* [11]. Next we give two other possible phase unwrapping solutions. We apply these three phase unwrapping techniques to our optimal reconstruction method.

2. DEFINITIONS

Our developments are based on the fourth order cumulant which never vanishes even under circularity hypothesis [12][13]. The fourth order cumulant of a complex random sequence $x(t)$ is given by :

$$C_4^x(t_1, t_2, t_3) = E\left\{x^*(\tau)x(\tau + t_1)x(\tau + t_2)x^*(\tau + t_3)\right\}$$
$$-E\left\{x^*(\tau)x(\tau + t_1)\right\}E\left\{x(\tau + t_2)x^*(\tau + t_3)\right\}$$
$$-E\left\{x^*(\tau)x(\tau + t_2)\right\}E\left\{x(\tau + t_1)x^*(\tau + t_3)\right\}$$
$$-E\left\{x(\tau + t_1)x(\tau + t_2)\right\}E\left\{x^*(\tau)x^*(\tau + t_3)\right\}, \quad (1)$$

and the corresponding fourth order spectrum [7][8] :

$$T_4^X(\omega_1, \omega_2, \omega_3) =$$
$$E\left\{X(\omega_1)X(\omega_2)X^*(-\omega_3)X^*(\omega_1 + \omega_2 + \omega_3)\right\}$$
$$-E\left\{X(\omega_1)X^*(\omega_1)\right\}E\left\{X(\omega_2)X^*(\omega_2)\right\}\delta(\omega_2 + \omega_3)$$
$$-E\left\{X(\omega_2)X^*(\omega_2)\right\}E\left\{X(\omega_1)X^*(\omega_1)\right\}\delta(\omega_1 + \omega_3)$$
$$-E\left\{X^*(\omega_3)X^*(-\omega_3)\right\}E\left\{X(\omega_1)X(-\omega_1)\right\}\delta(\omega_1 + \omega_2). \quad (2)$$

where $X(\omega)$ is the Fourier transform of the sequence $x(t)$.
• Remark : We draw the attention of the reader on the importance of the three planes appearing in (2).
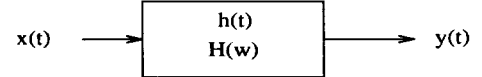

Figure 1: Identification scheme.

If the analyzed signal is the output of a LTI system driven by a non-gaussien zero-mean IID complex sequence $x(t)$ (Fig. 1), the fourth order spectrum phase satisfies:

$$\psi_4^Y(\omega_1, \omega_2, \omega_3) = \varphi^H(\omega_1) + \varphi^H(\omega_2) \quad (3)$$
$$-\varphi^H(-\omega_3) - \varphi^H(\omega_1 + \omega_2 + \omega_3) + k\pi,$$

where $\psi_4^Y(\omega_1, \omega_2, \omega_3)$ is the output trispectrum phase, $\varphi^H(\omega)$ is the system Fourier transform phase and $k = 0$ or 1 depending on the input kurtosis sign. However the input kurtosis sign beeing equal to the output kurtosis sign, the value of k is known from the output measurements, and (3) can be written :

$$\psi_4(\omega_1, \omega_2, \omega_3) = \psi_4^Y(\omega_1, \omega_2, \omega_3) - k\pi = \quad (4)$$
$$\varphi^H(\omega_1) + \varphi^H(\omega_2) - \varphi^H(-\omega_3) - \varphi^H(\omega_1 + \omega_2 + \omega_3).$$

In this paper, we use the fourth order statistics to reconstruct the transfer function phase only, since its magnitude can be obtained from the second order statistics.

3. FOURIER PHASE RECONSTRUCTION FROM THE TRISPECTRUM PHASE

3.1. A MULTIRESOLUTION RECONSTRUCTION METHOD

We give an extension of an algorithm developed in the third order case [10]:

$$\hat{\varphi}^H(0) = \hat{\varphi}^H(1) = \hat{\varphi}^H(2) = 0$$

(at the sampling frequency and its multiples)

For $n = 1, \ldots, \log_2 N$

$$
\left|
\begin{array}{l}
\hat{\varphi}^H\left(\frac{1}{2^n}\right) = \frac{1}{2}\left[
\begin{array}{l}
\psi_4\left(\frac{1}{2^n}, \frac{1}{2^n}, \frac{1}{2^{n-1}}\right) \\
+\hat{\varphi}^H\left(\frac{-1}{2^{n-1}}\right) + \hat{\varphi}^H\left(\frac{1}{2^{n-2}}\right)
\end{array}
\right] \quad (5) \\
\text{For } k = 0, \ldots, 2^{n-1} - 2 \\
\hat{\varphi}^H\left(\frac{2k+3}{2^n}\right) = -\psi_4\left(\frac{k}{2^{n-1}}, \frac{1}{2^n}, \frac{1}{2^{n-1}}\right) + \hat{\varphi}^H\left(\frac{k}{2^{n-1}}\right) \\
\qquad +\hat{\varphi}^H\left(\frac{1}{2^n}\right) - \hat{\varphi}^H\left(\frac{-1}{2^{n-1}}\right) \quad (6)
\end{array}
\right.
$$

Where $\hat{\varphi}^H(\omega)$ is the reconstructed Fourier phase and $\psi_4(\omega_1, \omega_2, \omega_3)$ is the estimation of the system trispectrum phase (cf. (4)).

- This algorithm requires no phase unwrapping.

3.2. LEAST SQUARES RECONSTRUCTION

The criterion and the general formula for real signals are given in [5]. This reconstruction requires a prior phase unwrapping (cf. [11]). In the complex case, the optimal fourth order solution is given by the phase minimizing:

$$\sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} \|\psi_4(\omega_1,\omega_2,\omega_3) - \hat{\psi}_4^H(\omega_1,\omega_2,\omega_3)\|^2, \quad (7)$$

where $\hat{\psi}_4^H(\omega_1,\omega_2,\omega_3) = \hat{\varphi}^H(\omega_1) + \hat{\varphi}^H(\omega_2) - \hat{\varphi}^H(-\omega_3) - \hat{\varphi}^H(\omega_1 + \omega_2 + \omega_3)$.
The minimum is obtained when

$$\hat{\varphi}^H(\omega) = \frac{1}{2N^2}\sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\psi_4(\omega,\omega_1,\omega_2)$$

$$-\frac{1}{2N^2}\sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\psi_4(\omega_1,\omega_2,-\omega) + K, \quad (8)$$

where K is an arbitrary constant. However, due to the divisions in (8), the value of $\psi_4(\omega_1,\omega_2,\omega_3)$ must be known in the interval $[-4\pi; 4\pi]$, but from the measurements, this value is known in $[-\pi; \pi]$. In the next paragraph, we give different algorithms to obtain the unwrapped phase value from its wrapped value.

3.2.1. PHASE UNWRAPPING ALGORITHMS

- Marron's bispectrum phase unwrapping algorithm

Marron et al. [11] have shown that it is possible to deduce all the N^2 unwrapped bispectrum phases $\psi_3(\omega_1,\omega_2)$ from the $N-1$ modulo 2π bispectrum phases $\psi_3(1,\omega)$, $\omega = 1, 2, \ldots, N-1$.
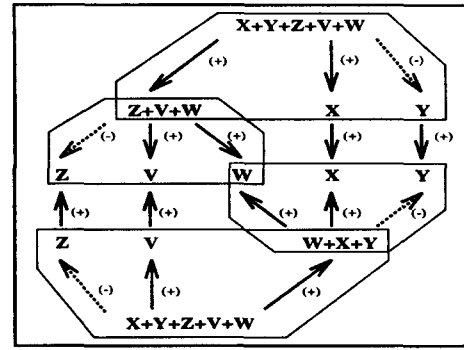

Figure 2: Scheme representing the relationship between four trispectrum phases. It illustrates eq.(10) when eq.(3) is satisfied.

To obtain the unwrapped phases, they use the following equation:

$$
\begin{aligned}
\psi_3(\omega_1,\omega_2) &= \psi_3(\omega_1 - 1, \omega_2 + 1) + \psi_3(1,\omega_2) \\
&- \psi_3(1,\omega_1 - 1). \quad (9)
\end{aligned}
$$

In the next hereunder, we extend the Marron's algorithm to the fourth order spectrum of complex signals.

- (α) Extension of Marron's algorithm for the fourth order spectrum

The fourth order extension of Marron's formula is represented in figure 2. Its expression is :

$$
\begin{aligned}
\psi_4(z + v + w, x, y) + \psi_4(w, v, z) &= \\
\psi_4(w, x, y) + \psi_4(w + x + y, v, z), \quad (10)
\end{aligned}
$$

for all v, w, x, y, z.

Note that this formula can be generalized to any order. It is always a four terms identity.
It is possible to deduce all the N^3 unwrapped trispectrum phases from the $(N - 1)$ modulo 2π trispectrum phases used in the multiresolution algorithm (see eq.5 and 6) using (10)[14].
The trispectrum phases deduced by this unwrapping procedure is compared to the measured modulo 2π trispectrum phase and a phase of $2q\pi$ is added to the measured phase so that their difference will be less than π.
There are other approaches for performing phase unwrapping. Here are two methods that are also efficient in practice.

- (β) Multiresolution used to unwrap the trispectrum phase

The multiresolution method gives a first approximation of the channel phases $\hat{\varphi}^H(\omega)$. Those values are then used to compute the trispectrum phases in the interval $[-4\pi, 4\pi]$ using equation (3). Another efficient solution consists in combining the multiresolution method with the optimal method as shown in the next paragraph.

- (γ) Multiresolution combined with the optimal method

Such a combination is possible thanks to the iterative structure of the multiresolution method. The multiresolution will be combined with the optimal method as follows :

- at step n of the algorithm, the multiresolution method gives a first estimate of $\hat{\varphi}^H\left(\frac{2m+1}{2^n}\right)$ for $m = 0, 1, ..., 2^{n-1} - 1$.

- these values, and those calculated in the previous steps (at lower resolutions), are used to unwrap the trispectrum phases : $\psi_4\left(\frac{p}{2^n}, \frac{q}{2^n}, \frac{r}{2^n}\right)$ for $p, q, r = 0, 1, ..., 2^n - 1$ using (4).

- Next, the LS estimation (8) uses these trispectrum phases to give an improved estimation of $\hat{\varphi}^H\left(\frac{2m+1}{2^n}\right)$ for $m = 0, 1, ..., 2^{n-1} - 1$.

- Finally, these last estimates will be used to initialize the next step $(n+1)$ of the algorithm.

4. IDENTIFICATION AND KURTOSIS MAXIMIZATION

In this section, we show that the optimal least-squares identification is very similar to the well known kurtosis maximization.

4.1. KURTOSIS MAXIMIZATION CRITERION

In this section, we use the scheme used in the equalization context (see Fig.3).
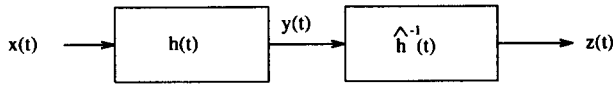The criterion was proposed by D. Donoho and later by

Figure 3: Equalization scheme.

O. Shalvi and E. Weinstein [15] [16] [17] in order to recover the input sequence $x(t)$. It consists in estimating $\hat{h}^{-1}(t)$ through the maximization of $|K(z)|$ under the constraint $E\left\{|z|^2\right\} = E\left\{|x|^2\right\}$, where $K(z)$ is the kurtosis of z.
In a first step O. Shalvi and E. Weinstein propose to whiten the output signal so that they are essentially reconstructing the channel Fourier transform phase just like the reconstruction algorithms in the frequency domain.

4.2. EXPRESSION OF THE KURTOSIS MAXIMIZATION IN THE FREQUENCY DOMAIN

The kurtosis of the output sequence $z(t)$ is given by :

$$K(z) = C_4^z(0,0,0) = \sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} T^Z(\omega_1, \omega_2, \omega_3)$$

$$= \sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} \left|T^Z(\omega_1, \omega_2, \omega_3)\right| e^{j\psi_4^Z(\omega_1,\omega_2,\omega_3)}. \quad (11)$$

Since $K(z)$ is a real number, its modulus is given by :

$$|K(z)| = \sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} \left|T^Z(\omega_1, \omega_2, \omega_3)\right| e^{j\left(\psi_4^Z(\omega_1,\omega_2,\omega_3) - k\pi\right)}, \quad (12)$$

where $k = 1$ if $K(z) < 0$ and $k = 0$ if $K(z) \geq 0$.
• If the output is whitened, $\left|T^Z\right|$ is constant.
• $K(z)$ being a real number, the complex exponentials in (12) are replaced by their real part.

Then the Fourier phase maximizing the kurtosis of the output is the phase which maximizes :

$$J = \frac{|K(z)|}{|T^Z|} = \quad (13)$$

$$\sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} \cos\left[\psi_4^Y(\omega_1, \omega_2, \omega_3) - \hat{\psi}_4^H(\omega_1, \omega_2, \omega_3) - k\pi\right].$$

• Since the kurtosis sign of z is the kurtosis sign of x, the value of k is the same as in (3); then we can replace $\psi_4^Y(\omega_1, \omega_2, \omega_3) - k\pi$ by $\psi_4(\omega_1, \omega_2, \omega_3)$ (cf. 4). Finally, (13) becomes :

$$J = \sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} \cos\left[\psi_4(\omega_1, \omega_2, \omega_3) - \hat{\psi}_4^H(\omega_1, \omega_2, \omega_3)\right]. \quad (14)$$

4.3. TAYLOR EXPANSION OF THE KURTOSIS

If the trispectrum is factorizable and its phase is accurately estimated, the difference between $\psi_4^Y(\omega_1, \omega_2, \omega_3)$ and $\hat{\psi}_4^H(\omega_1, \omega_2, \omega_3)$ will be small and we can expand eq. (14) :

$$J = \sum_{\omega_1=0}^{N-1}\sum_{\omega_2=0}^{N-1}\sum_{\omega_3=0}^{N-1} \left[1 - \frac{1}{2}\left[\psi_4(\omega_1, \omega_2, \omega_3) - \hat{\psi}_4^H(\omega_1, \omega_2, \omega_3)\right]^2 \quad (15)\right.$$
$$\left. + \frac{1}{24}\left[\psi_4(\omega_1, \omega_2, \omega_3) - \hat{\psi}_4^H(\omega_1, \omega_2, \omega_3)\right]^4 + ...\right].$$

If we limit the development to the second term, the maximization of this criterion reduces to the minimization of the LS criterion obtained in the frequency domain (7). Under this hypothesis, kurtosis maximization reduces to the minimization of a quadratic criterion for which we know the analytic solution (8). However we introduce the phase unwrapping problem which has a solution if the trispectrum is factorizable and the trispectrum phase estimate accurate enough.

5. SIMULATION RESULTS

We have simulated a channel using the 26th order complex FIR filter proposed in [18] deduced from experimental data. The input was a 4-QAM IID signal. Note that the input kurtosis is negative.
Figure 4 shows the analyzed channel frequency response modulus. Figure 5 shows the results of the multiresolution method alone (trispectrum averages on 50000 sequences). Figure 6 shows the results of the optimal method using the extension of Marron (α) and the combination (γ) unwrapping methods (trispectrum averages on 10000 sequences). The phase unwrapping methods α and β give comparable results while γ improves slightly the results. As expected, the multiresolution algorithm needs a very accurate estimation of the trispectrum to reconstruct an acceptable solution, consequently it requires a large amount of data. But the optimal solution is acceptable even if the variance on the estimated trispectrum is large, provided the trispectrum phase is correctly unwrapped.
In the other hand, from the simulation results, we can make the following interesting conclusion : if the phase unwrapping step is correct, the number of samples required to obtain a good quality identification is drastically reduced.
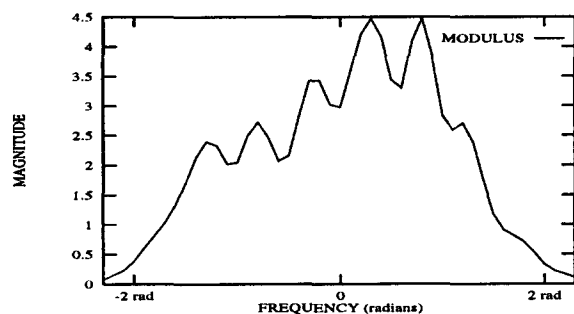
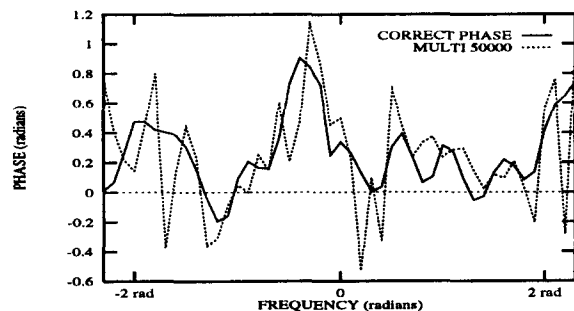Figure 4: Fourier transform modulus : $|H(\omega)|$.

Figure 5: Fourier transform phase reconstruction with the multiresolution after 10000 iterations.
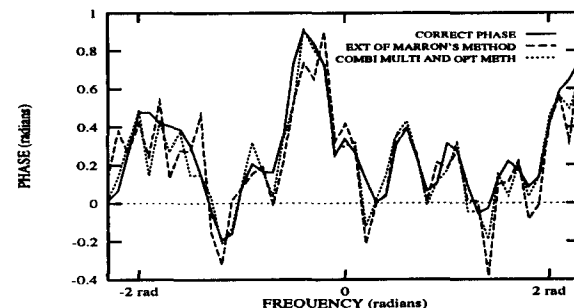

Figure 6: Fourier transform phase reconstruction with the optimal method after 10000 iterations.

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