DISTINCTION BETWEEN POLYNOMIAL PHASE SIGNALS WITH CONSTANT AMPLITUDE AND RANDOM AMPLITUDE

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ABSTRACT

In this paper we propose to distinguish between constant amplitude polynomial phase signals and the ones having random amplitude. We study four possibilities for the modulating process. We show that the distinction of this kind of signals is not always possible when using the Polynomial Phase Transform. In fact, in some applications, we show that we cannot estimate the phase of the signal with this transform. In order to solve this problem, we introduce a new transform which allows us to estimate this phase in these particular situations. The obtained transform is referred to as the Modified Polynomial Phase Transform.

1. INTRODUCTION

Polynomial phase signals (PPS) are commmonly used in many areas such as radar, sonar and communication. Many techniques, based on higher-order statistics, such as the Polynomial Wigner-Ville Distribution (PWVD) [1], the Polynomial Phase Transform (PPT) [2] and the Generalized Ambiguity Function (GAF) [3], have been introduced to process such signals. The principle of the two last transforms is to nonlinearly transform the signal to obtain a sinusoid at a certain frequency which is directly related to the higher-order coefficient of the phase of the signal.

In this paper we consider the following model

$$y(n) = b(n)e^{j\phi(n)} + w(n), \quad n = 1, ..., N_e \qquad (1)$$

with

$$\phi(n) = \sum_{i=0}^{N} a_i (n\Delta)^i \qquad (2)$$

where Δ is the sampling period, $w(n)$ is a complex circular $\mathcal{N}(0, \sigma_w^2)$ white noise and $b(n)$ is a real modulating process independent of $w(n)$.

In [4], the estimation of the deterministic time-varying amplitude PPS was addressed. The estimation of the random amplitude PPS, using the cyclostationary approach, was studied in [5]. In [6], the authors proposed the analysis of FM signals affected by a Gaussian noise using the Reduced Wigner-Ville Trispectrum (RWVT). In [7], the author addressed the problem of distinguishing between random amplitude harmonics and constant amplitude harmonics. In this paper, we study the problem of differentiating between PPS with a random amplitude and with a constant one. We consider the following situations for the real modulating process $b(n)$

- M1-$b(n) = b_0 \neq 0$.
- M2-$b(n) = 0$.
- M3-$b(n)$: white zeromean, with finite moments.
- M4-$b(n)$: white nonzeromean, with finite moments.

2. PROBLEM DESCRIPTION

The PPT [2] and the GAF [3], were introduced to process constant amplitude PPS. It has been shown in [3] that the two transforms lead to the same conclusions for the estimation of polynomial phase coefficients. In this paper, the proposed analysis uses the PPT. However, all the results which will be obtained through this analysis are valid for the GAF. Let us recall the definition of the PPT defined, in [2], by

$$\mathcal{P}_N(y, \theta, \tau) = \sum_{n=N_i}^{N_e} \prod_{k=0}^{N-1} \left(y^{(*k)}(n - k\tau) \right)^{c_k} e^{-jn\Delta\theta} \qquad (3)$$

where $c_k = \binom{N-1}{k}$ and $N_i = 1 + (N-1)\tau$.
The main result of the PPT is that for a constant amplitude PPS $y(n) = b_0 e^{j\phi(n)}$ we have the following result

$$a_N = \frac{1}{N!(\Delta\tau)^{N-1}} \arg \max_{\theta} |\mathcal{P}_N(y, \theta, \tau)| . \qquad (4)$$

Let us now discuss the estimation of the higher-order coefficient a_N of the phase ϕ of the equation (1). In this order, let us evaluate the expectation $E[\mathcal{P}_N(y,\theta,\tau)]$. It is then possible to establish the following results for the four situations of $b(n)$.
For the model M1

$$E[\mathcal{P}_N(y,\theta,\tau)] = b_0^{2^{N-1}}\mathcal{P}_N(e^{j\phi(n)},\theta,\tau). \quad (5)$$

For the model M2

$$E[\mathcal{P}_N(y,\theta,\tau)] = 0. \quad (6)$$

For the model M3

$$E[\mathcal{P}_N(y,\theta,\tau)] = 0. \quad (7)$$

For the model M4

$$E[\mathcal{P}_N(y,\theta,\tau)] = \left(\prod_{k=0}^{N-1} m_{c_k}\right)\mathcal{P}_N(e^{j\phi(n)},\theta,\tau) \neq 0 \quad (8)$$

where $m_{c_k} = E[b^{c_k}(n)]$.

From the results (5), (6), (7) and (8) different observations can be made:
1). For PPS with constant amplitude and the ones with white nonzeromean amplitude, $|E[\mathcal{P}_N(y,\theta,\tau)]|$ produces a spectral line at $N!a_N(\Delta\tau)^{N-1}$, i.e a_N is given by

$$a_N = \frac{1}{N!(\Delta\tau)^{N-1}}\arg\max_{\theta}|E[\mathcal{P}_N(y,\theta,\tau)]|. \quad (9)$$

Thus, we see that the determination of the coefficient a_N is possible via (9). However, if we do not have any prior on the process $b(n)$, it will be impossible to distinguish between the nature of the two amplitudes.
2). From equations (6) and (7), we have for the models M1 and M2, $E[\mathcal{P}_N(y,\theta,\tau)]=0$. So it will be impossible to estimate the coefficient a_N and thus the phase ϕ, and it will, also, be impossible to distinguish between the purely stationary case, $y(n) = w(n)$, and a PPS with white zeromean amplitude . For this purpose, we should introduce another technique which allows us to estimate the phase in the case of model M3.

3. DISITINGUISHING BETWEEN PPS WITH CONSTANT AND WHITE NONZEROMEAN AMPLITUDE

The problem of distinction between both cases is possible using the following remarks. In fact theoretically after the determination of the coefficients $\{a_i\}$ and multiplying the original signal by $\exp(-j\phi(n))$, we obtain respectively for the two models M1 and M4

$$\tilde{z}_1(n) = b_0 + w(n)e^{-j\phi(n)} \quad (10)$$
$$\tilde{z}_2(n) = b(n) + w(n)e^{-j\phi(n)}. \quad (11)$$

Furthermore, it is possible to show that

$$E[\mathcal{P}_1((\tilde{z}_1(n)-b_0)^2,\theta,\tau)] = 0 \quad (12)$$
$$E[\mathcal{P}_1((\tilde{z}_2(n)-m_b)^2,\theta,\tau)] = \mathcal{P}_1(\sigma_b^2,\theta,\tau) \quad (13)$$

where $\sigma_b^2 = E[b^2(n)] - m_b^2$, $m_b = m_1$ and $\mathcal{P}_1(x,\theta,\tau)$ is the DFT of the signal $x(n)$.

Thus from (12) and (13) we can distinguish between M1 and M4. In fact $|E[\mathcal{P}_1((\tilde{z}_1(n)-b_0)^2,\theta,\tau)]|=0$, $\forall\theta$, for a PPS with constant amplitude whereas $|E[\mathcal{P}_1((\tilde{z}_2(n)-m_b)^2,\theta,\tau)]|$ will produce a peak at the origin for a PPS with white nonzeromean amplitude. We obtain the following algorithm

- Evaluate $|E[\mathcal{P}_N(y,\theta,\tau)]|$.
  - If there is no peak, then M2 and M3 are possible. Stop.
  - Else M1 and M4 are possible. Continue.
- Estimate $\{\hat{a}_i\}$ and $\hat{b}$ by the PPT-based algorithm.
  - In the case of the PPS with constant amplitude $\hat{b} = \hat{b}_0 \approx b_0$.
  - In the case of the PPS with white nonzeromean amplitude $\hat{b} = \hat{m}_b \approx m_b$
- Apply $|E[\mathcal{P}_N((ye^{-j\sum_{i=0}^{N}\hat{a}_i(n\Delta)^i} - \hat{b})^2,\theta,\tau)]|$
  - If there is no peak, then M1 holds. Stop.
  - Else M2 holds. Stop.

Example1
We generate N_e=300 samples of a third-order PPS with a_0=1.0003, a_1=7.8540, a_2=0.1257 et a_3=0.0063. The sampling period was Δ=0.2s. The parameter τ is chosen such that it takes at each step of the PPT-based algorithm $\tau=\tau_m=\frac{N_e}{m}$ [2]. In the model M1, b_0=1 whereas in M2 the process $b(n)$ is $\mathcal{N}(0.5,1)$. A complex circular $\mathcal{N}(0,\sigma_w^2)$ white noise is added to $b(n)e^{j\phi(n)}$. The signal to noise ratio (SNR) is defined in the first case by $\text{SNR}=10\log_{10}(b_0^2/\sigma_w^2)$ and in the second case by $\text{SNR}=10\log_{10}(\sigma_b^2/\sigma_w^2)$. In this example and in the following one we take SNR=15 dB. $E[\mathcal{P}_N]$ is estimated by averaging $\mathcal{P}_N$ of 100 independent realizations.
From the peaks appearing in Fig.1(a), Fig.1(b), Fig.1(c), Fig.2(a), Fig.2(b) and Fig.2(c), it appears that the values of the estimators $\hat{a}_3$, $\hat{a}_2$ and $\hat{a}_1$ and then $\hat{a}_0$ are almost the same for the two models M1 and M4. These values are respectively equal to 0.0063, 0.1257, 7.8540 and 0.9998. The set of these estimate values are very close to the real ones. However for the estimation of $\hat{b}$, we found for the first model M1 $\hat{b}_0 = 1.0007 \approx b_0 \triangleq 1$ and for M4 $\hat{m}_b = 0.4981 \approx m_b \triangleq 0.5$. In order to distinguish between the two amplitudes we evaluate $|E[\mathcal{P}_1((y\exp(-j\sum_{i=0}^{3}\hat{a}_i(n\Delta)^i) - \hat{b})^2,\theta,\tau)]|$.

Fig.1(d) does not reveal any peak while Fig.2(d) shows a peak at the origin corresponding to the variance of the process $b(n)$. Thus, we conclude that the first simulation corresponds to M1 and the second one to M4.

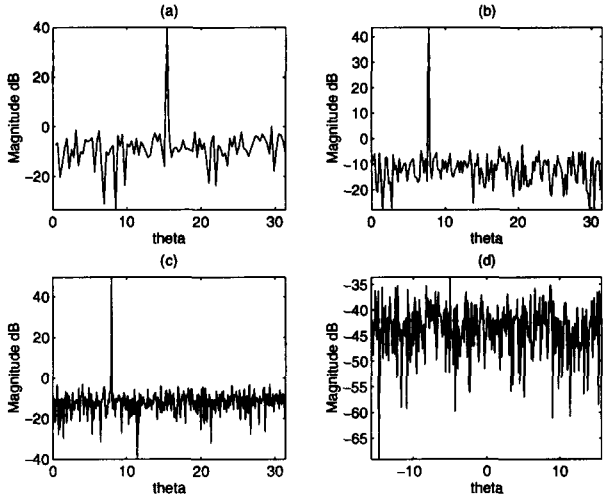

Figure 1: (a)-$|E[\underline{\mathcal{P}}_3(y,\theta,\tau)]|$, (b)-$|E[\underline{\mathcal{P}}_2(y,\theta,\tau)]|$ after removing $\hat{a}_3$, (c)-$|E[\underline{\mathcal{P}}_1(y,\theta,\tau)]|$ after removing $\hat{a}_3$ and $\hat{a}_2$, (d)-$|E[\underline{\mathcal{P}}_1((y\exp(-j\sum_{i=0}^{3}\hat{a}_i(n\Delta)^i)-\hat{b})^2,\theta,\tau)]|$.

4. THE MODIFIED PPT

When the application of the PPT does not reveal any peak, two models of the signal are possible, that is the PPS with white zeromean amplitude and the purely stationary case corresponding to $y(n) = w(n)$. Furthermore, in the first situation the estimation of ϕ using the PPT is impossible. It is therefore necessary to introduce another technique which allows us, in this case, to both estimate ϕ and distinguish between the PPS with white zeromean amplitude and the purely stationary case. For this purpose, we can for example square the kernel of the PPT. This operation leads to a new kernel where only the even-order moments of $b(n)$ appear and the first-order moment, which is at the origin of the result (7), disappears. The resulting distribution referred to as the Modified-PPT (MPPT) and denoted by $\underline{\mathcal{M}}_N(y,\theta,\tau)$, is defined by

$$\underline{\mathcal{M}}_N(y,\theta,\tau) \triangleq \sum_{n=N_i}^{N_e}\prod_{k=0}^{N-1}\left(y^{(*k)}(n-k\tau)\right)^{2c_k} e^{-jn\Delta\theta} \tag{14}$$

The following special cases illustrate this definition

$$\underline{\mathcal{M}}_1(y,\theta,\tau) = \sum_{n=1}^{N_e} y^2(n)e^{-jn\Delta\theta}$$

$$\underline{\mathcal{M}}_2(y,\theta,\tau) = \sum_{n=1+\tau}^{N_e} y^2(n)\left(y^*(n-\tau)\right)^2 e^{-jn\Delta\theta}.$$

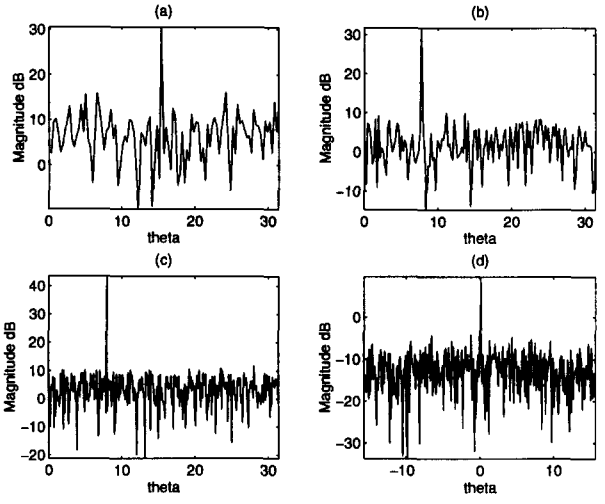

Figure 2: (a)-$|E[\underline{\mathcal{P}}_3(y,\theta,\tau)]|$, (b)-$|E[\underline{\mathcal{P}}_2(y,\theta,\tau)]|$ after removing $\hat{a}_3$, (c)-$|E[\underline{\mathcal{P}}_1(y,\theta,\tau)]|$ after removing $\hat{a}_3$ and $\hat{a}_2$, (d)-$|E[\underline{\mathcal{P}}_1((y\exp(-j\sum_{i=0}^{3}\hat{a}_i(n\Delta)^i)-\hat{b})^2,\theta,\tau)]|$.

It is easy to see that according to the definitions of the MPPT and the PPT, the two operators applied to an FM signal, $b_0e^{j\phi(n)}$, are related to each other since

$$\underline{\mathcal{M}}_N(b_0e^{j\phi(n)},\theta,\tau) = b_0^{2^N}\underline{\mathcal{P}}_N(e^{j2\phi(n)},\theta,\tau). \tag{15}$$

Thus for a phase ϕ given by (2), we obtain

$$a_N = \frac{1}{2N!(\tau\Delta)^{N-1}}\arg\max_{\theta}|\underline{\mathcal{M}}_N(y,\theta,\tau)|. \tag{16}$$

Let us now evaluate $E[\underline{\mathcal{M}}_N(y,\theta,\tau)]$ and discuss the estimation of the phase ϕ appearing in (1) for the models M2 and M3. It is then possible to show, for the model M2 corresponding to the purely stationary case, the following result

$$E\left[\underline{\mathcal{M}}_N(y,\theta,\tau)\right] = 0 \tag{17}$$

while for the model M3, we obtain

$$E\left[\underline{\mathcal{M}}_N(y,\theta,\tau)\right] = \left(\prod_{k=0}^{N-1} m_{2c_k}\right)\underline{\mathcal{P}}_N(e^{j2\phi(n)},\theta,\tau) \neq 0. \tag{18}$$

Thus from equations (17) and (18), we see that for the purely stationary case, $y(n) = w(n)$, $E\left[\underline{\mathcal{M}}_N(y,\theta,\tau)\right]$ vanishes $\forall\theta$, while for a PPS with white zeromean amplitude, $|E\left[\underline{\mathcal{M}}_N(y,\theta,\tau)\right]|$ produces a spectral line at $2N!a_N(\Delta\tau)^{N-1}$, i.e a_N is given by

$$a_N = \frac{1}{2N!(\Delta\tau)^{N-1}}\arg\max_{\theta}|E\left[\underline{\mathcal{M}}_N(y,\theta,\tau)\right]|. \tag{19}$$

The determination of the higher-order coefficient a_N is therefore possible when using the MPPT. Consequently, all the other coefficients will be estimated using the MPPT instead of the PPT. Hence the following algorithm of differentiating and estimation of the phase ϕ of the signal

- Evaluate $|E[\underline{\mathcal{P}}_N(y,\theta,\tau)]|$.
  - If there is a peak, then M1 and M4 are possible. Stop.
  - Else M2 and M3 are possible. Continue.
- Evaluate $|E[\underline{\mathcal{M}}_N(y,\theta,\tau)]|$.
  - If there is no peak, then M2 holds. Stop.
  - Else the model M3 holds. Continue the estimation of $\{a_i\}$ using the MPPT.

Example 2
We consider the same signal and we consider $b(n)$ $\mathcal{N}(0,1)$. Fig.3(a) presents $|E[\underline{\mathcal{P}}_3(y,\theta,\tau)]|$. It does not

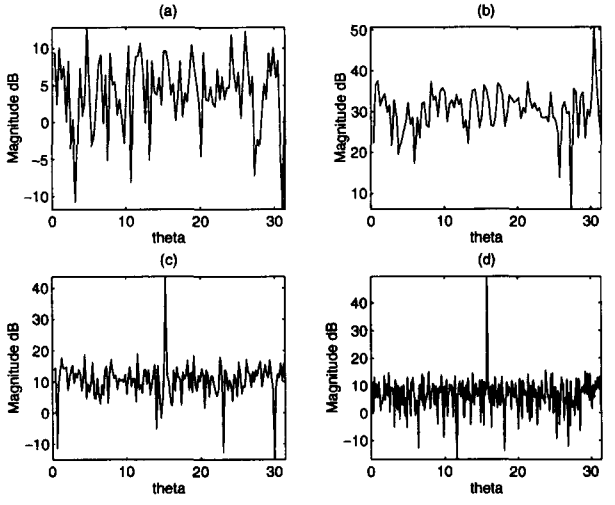

Figure 3: (a)-$|E[\underline{\mathcal{P}}_3(y,\theta,\tau)]|$, (b)-$|E[\underline{\mathcal{M}}_3(y,\theta,\tau)]|$, (c)-$|E[\underline{\mathcal{M}}_2(y,\theta,\tau)]|$ after removing $\hat{a}_3$, (d)-$|E[\underline{\mathcal{M}}_1(y,\theta,\tau)]|$after removing $\hat{a}_3$ and $\hat{a}_2$.

show any distinct peak, which is in accordance with the theoretical result (7). However the application of $|E[\underline{\mathcal{M}}_3(y,\theta,\tau)]|$ reveals a peak around 30.24, which is the value of $12a_3(\Delta\tau)^2$, as shown in Fig.3(b). Thus the estimated value of a_3 is $\hat{a}_3$ 0.0063. In Fig.3(c), we show $|E[\underline{\mathcal{M}}_2(y,\theta,\tau)]|$ after removing $\hat{a}_3$. This figure shows a spectral line around 15.084, which is the value of $4a_2\Delta\tau$. We find for $\hat{a}_2$ 0.1257 and in Fig.1(d), we observe a peak around 15.708 which is $2a_1$. The estimated value of a_1 is 7.854. The estimated value $\hat{a}_0$ of a_0 is 1.0003.

5. CONCLUSION

In this paper, we studied the problem of distinguishing between PPS with constant amplitude and random amplitude PPS. We considered four situations for the modulating process. When the application of the expectation of the PPT reveals peaks, two situations are possible: either the signal is a PPS with constant amplitude or the signal is a PPS affected by a multiplicative white nonzeromean process. We proposed in this paper one solution to distinguish between these two cases. However, when we are confronted with an absence of a spectral line, two situations are possible: either the purely stationary case or a PPS affected by a white zeromean multiplicative process, furthermore, in this last case, we showed that the PPT does not allow us to estimate the phase. We proposed, so, a solution to this problem by introducing a modified version of the PPT. In addition, this last transform allows us distinction between the two models M2 and M3.

6. REFERENCES

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