PERFECT RECONSTRUCTION CIRCULAR CONVOLUTION FILTER BANKS AND THEIR APPLICATION TO THE IMPLEMENTATION OF BANDLIMITED DISCRETE WAVELET TRANSFORMS

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ABSTRACT

This paper, introduces a new filter bank structure called the *perfect reconstruction circular convolution* (PRCC) filter bank. These filter banks satisfy the perfect reconstruction properties, namely, the paraunitary properties in the *discrete frequency* domain. We further show how the PRCC analysis and synthesis filter banks are completely implemented in this domain and give a simple and a flexible method for the design of these filters. Finally, we use this filter bank structure for a frequency sampled implementation of the discrete wavelet transform based on orthogonal bandlimited scaling functions and wavelets.

1 INTRODUCTION

In this paper we present a new multirate filter bank structure which we call *perfect reconstruction circular convolution* (PRCC) filter bank. We further develop simple and flexible methods for designing these filters to specification.

The formulation of this new filter bank structure has been motivated in part by the search for efficient methods to implement the discrete wavelet transform (DWT) based on orthogonal bandlimited scaling functions and wavelets. A considerable amount of research has been done in the area of wavelets that are compactly supported in time. However, there are situations where a bandlimited scaling function or wavelet could be more appropriate. Relevant examples can be found in a variety of fields such as communication, signal analysis and pattern recognition [2, 5, 7]. We provide another example of such a situation in section 3. Bandlimited wavelets and scaling functions have several interesting properties. For example, they provide an easy solution set to the problem of designing orthonormal multiresolution decomposition, generating wavelets that are matches to arbitrarily specified signals [5, 4]. Using such wavelets Samar et. al. [7] have shown superior convergence of multiresolution representations for bandlimited wavelets as compared to wavelets with compact time support for EEG data.

An impediment to more widespread use of bandlimited wavelets has been their infinite time support that makes the corresponding filters of infinite impulse response (IIR) type, usually without a finite order difference equation. To get around this, an appropriately truncated version of the time response can be used. This results in loss of the bandlimitedness property. For the DWT, it also means a loss of invertibility and perfect reconstruction. Here, we introduce a filter bank structure that provides a framework for a frequency sampled implementation of bandlimited scaling functions and wavelets while guaranteeing perfect reconstruction at the same time.

The paper is organized as follows. Section 2 describes the Meyer scaling function. Section 3 illustrates a scenario where the Meyer scaling function or its generalization could be an optimal choice. Section 4 describes the PRCC framework and presents a simple and flexible method for the design of these filters. Section 5 explains how the PRCC framework could be used for the frequency sampled implementation of bandlimited wavelet transform. Section 6 explains the symmetric extension implementation of the PRCC filter banks to reduce edge effects. Finally, section 7 presents the conclusion.

2 THE MEYER SCALING FUNCTION

As mentioned above, in this paper, we will show how the PRCC filter bank structure can be used for a frequency sampled implementation of the DWT based on orthogonal scaling functions and wavelets. It has been shown that a generalized version of the Meyer class of scaling functions are the only bandlimited functions which define a orthogonal multiresolution analysis [4]. In other words, all orthogonal bandlimited scaling functions and wavelets belong to a generalized version of the Meyer class.

The Meyer scaling function, $\phi(t)$ satisfies the following properties [4]:

1. The spectrum of $\phi(t)$, $\Phi(\omega)$ is bandlimited to $|\omega| = 4\pi/3$.
2. $|\Phi(\omega)| = 1$ for $|\omega| \leq 2\pi/3$.
3. $|\Phi(\pi-\omega)|^2 + |\Phi(\pi+\omega)|^2 = 1$ for $|\omega| \leq \pi/3$.
4. The Poisson sum, $\sum_k |\Phi(w+2\pi k)|^2 = 1$. This is equivalent to $\langle \phi(t), \phi(t-n)\rangle = \delta(n)$ In other words, the Meyer scaling function is orthogonal to its integer translates.

3 MOTIVATION

This section illustrates a situation where the Meyer scaling function could naturally arise in the context of sampling a bandlimited function. Consider the system shown in figure 1. This corresponds to an *approximation sampling procedure*. Here, $a(t)$ is an anti-aliasing filter. This is followed by a *unit-sampler* and $b(t)$ is the reconstruction filter. We now ask the following questions: Given a signal $f(t)$, does there exist an optimal pair $a(t)$, $b(t)$, which minimizes the mean square error between the original signal $f(t)$ and its approximation $f_a(t)$? If yes, what are the properties that this pair satisfies?

Now, from Figure 1,

$$g(n) = \int f(\tau)a(n-\tau)d\tau \qquad (1)$$

This corresponds to unit sampling of $g(t)$. Using (1) we have

$$f_a(t) = \sum_n g(n)b(t-n) \qquad (2)$$

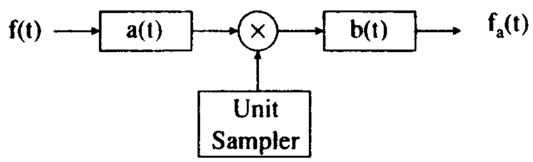


Figure 1. Approximation sampling block diagram.

Minimizing the mean square error between $f(t)$ and $f_a(t)$ with respect to $b(t)$ gives

$$B(\omega) = \frac{F(\omega)}{\sum_k F(\omega + 2\pi k)A(\omega + 2\pi k)} \quad (3)$$

where $B(\omega)$ is the Fourier transform of $b(t)$ and $A(\omega)$ and $F(\omega)$ are Fourier transforms of $a(t)$ and $f(t)$ respectively. Equation (3) thus represents the optimal reconstruction filter given $f(t)$ and $a(t)$. Obviously, there are certain conditions that both $A(\omega)$ and $F(\omega)$ must satisfy to ensure the existence of $B(\omega)$. For example, for $B(\omega)$ to exist, we require the denominator of (3) to be non-zero at all ω. With this condition, the Poisson sum

$$\sum_n A(\omega + 2\pi n)B(\omega + 2\pi n) = 1 \quad (4)$$

In words, $a(-t)$ is orthogonal to the non-zero interger translates of $b(t)$. Now, taking the Fourier transform of (2) and using (3) we have

$$F_a(\omega) = F(\omega) \quad (5)$$

This result implies that it is possible to design $B(\omega)$ to exactly reconstruct $F(\omega)$ from its filtered samples provided the denominator in (3) does not go to zero for all ω. The above equations could have interesting consequences which need to be investigated further. For now, we use these equations to illustrate two specific cases which serve as our motivation.

Let $F(\omega)$ be bandlimited to $|\omega| = \pi$. Then, (3) gives

$$\begin{aligned} A(\omega)B(\omega) &= 1 \qquad |\omega| \leq \pi \\ &= 0 \qquad \text{otherwise} \end{aligned} \quad (6)$$

This suggests that $A(\omega)$ and $B(\omega)$ are spectral factors of the ideal brickwall filter and corresponds to a generalization of the classical sampling theorem. For example, if $A(\omega) = 1$ then $B(\omega)$ corresponds to the ideal brickwall filter. That is, $a(t) = \delta(t)$ and $b(t) = \sin(\pi t)/\pi t$, the ideal interpolation filter.

Now consider the case when $F(\omega)$ and hence $F(\omega)A(\omega)$ are bandlimited to $|\omega| = 4\pi/3$. Sampling this signal at a rate of 1 corresponds to undersampling the signal and leads to aliasing. From 3 we have

$$A(\omega)B(\omega) = 1 \qquad |\omega| \leq 2\pi/3 \quad (7)$$

and

$$A(\pi - \alpha)B(\pi - \alpha) + A^*(\pi + \alpha)B^*(\pi + \alpha) = 1 \quad (8)$$

where $\alpha < \pi/3$. If we impose an additional constraint that $F(\omega)$ be real, we have

$$A(\pi - \alpha)B(\pi - \alpha) + A(\pi + \alpha)B(\pi + \alpha) = 1 \quad (9)$$

From section 2, (9) along with (7) and (4) imply that $A(\omega)$ and $B(\omega)$ are real spectral factors of $|\Phi(\omega)|^2$, the power spectrum of the Meyer scaling function. This discussion illustrates an example where the Meyer type of scaling function could be used to derive an optimal interpolator. The idea of using scaling functions as antialiasing and reconstruction filters in an approximation sampling system occurs in [1]. With this motivation, we introduce the idea of prcc filter banks in the next section.

4 PRCC FILTER BANKS

Before we describe the PRCC filter banks, we interpret the operations of downsampling and upsampling in terms of the discrete Fourier transform (DFT). Also, the input to the PRCC filter bank is a finite signal $x(n)$ of length N. Note that, to represent the DFT of a sequence we use the notations $X(e^{j2\pi k/N})$ and $X(k)$ interchangeably.

4.1 Downsampling

The input to the downsampler is a sequence $x(n)$ of N samples. The output of the downsampler, denoted by $y(n)$, is given by[8]

$$y(n) = x(Mn) \quad (10)$$

where the sequence $y(n)$ has N/M samples. It is assumed that N is an integer multiple of M.

The DFT of the output $Y(e^{j2\pi k/(N/M)})$ is given by [3]

$$Y(e^{j2\pi k/(N/M)}) = \frac{1}{M}\sum_{l=0}^{M-1} X(e^{j2\pi(k-\frac{N}{M}l)/N}) \quad (11)$$

$k = 0, 1, \ldots, \frac{N}{M} - 1$. Note that $Y(k)$ is N/M periodic. In other words, we take the IDFT of the first N/M points of the N point summation on the right hand side of (11) to get the downsampled signal. Thus, from (11), each coefficient of the DFT of the downsampled sequence $Y(k)$ is a sum of M coefficients of the DFT of the input sequence $X(k)$, spaced N/M samples apart. For example, when $M = 2$, the steps involved are the following:

- Take the DFT of $x(n)$
- Add the DFT of $x(n)$ and its N/2 rotated version. This makes use of the N periodicity of $X(k)$.
- Divide the resulting sequence by 2.
- Take the IDFT of the first N/2 samples.

This creates the $N/2$ point dowsampled sequence $y(n)$.

4.2 Upsampling

An L-fold upsampler takes an input sequence $x(n)$ and produces an output sequence defined as follows [8]:

$$y(n) = \begin{cases} x(n/L) & n = \text{multiple of } L \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

The output sequence has NL samples. The DFT of $y(n)$ in terms of the DFT of $x(n)$ is [3]

$$Y(e^{j2\pi k/(NL)}) = X(e^{j2\pi k/N}), \quad k = 0, 1, \ldots, NL - 1 \quad (13)$$

In words, the NL length DFT of the upsampled sequence $Y(e^{j2\pi k/(NL)})$ is nothing but a concatenation of L DFTs of $x(n)$.

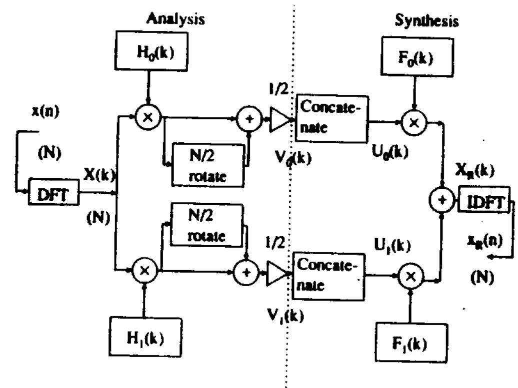


Figure 2. Perfect reconstruction circular convolution analysis and synthesis filter banks.

4.3 The Basic Procedure

PRCC filter banks are filter banks designed and implemented completely in the *discrete frequency domain.* In other words, these filter banks satisfy the conditions for perfect reconstruction over a discrete set of frequencies and the operations of downsampling, upsampling and filtering are carried out entirely in the discrete frequency domain. The basic procedure for their implementation can be understood by referring to the Figure 2. Here, we first take the DFT of the N length input signal $x(n)$. Next, we multiply this DFT, $X(k)$, pointwise with a sequence $H_0(k)$, which is the DFT of an N length sequence $h_0(n)$. This amounts to circularly convolving sequences, $h_0(n)$ and $x(n)$. The resultant sequence is then downsampled by two as explained in subsection 4.1. The procedure is repeated for the lower branch with $H_1(k)$, the DFT of $h_1(n)$. In this manner, we decompose the input sequence into two sequences of length $N/2$ whose DFTs we denote by $V_0(k)$ and $V_1(k)$ respectively. To recombine the two sequences, we upsample the sequences $V_0(k)$ and $V_1(k)$ as explained in subsection 4.2. This gives us two sequences $U_0(k)$ and $U_1(k)$ of length N. These are then multiplied pointwise with $F_0(k)$ and $F_1(k)$ which are the DFTs of N length sequences $f_0(n)$ and $f_1(n)$ respectively. They are the synthesis filters corresponding to the analysis filters $h_0(n)$ and $h_1(n)$ respectively. The output of the synthesis filter bank is thus given by $X_R(k) = F_0(k)U_0(k) + F_1(k)U_1(k)$. The reconstructed signal is then the inverse DFT (IDFT) of $X_R(k)$, namely, $x_R(n)$.

Note that, the PRCC filter bank is a framework based in the discrete frequency and is different from the work proposed by several authors before on fast implementation of FIR filter banks based on the FFT [6]. We now present the conditions for perfect reconstruction for the PRCC banks in the next subsection.

4.4 Conditions for perfect reconstruction

Since the input sequence is of length N it follows from the previous subsection that the filters also have a support of N samples. Furthermore, we assume that all sequences and filters are *real valued.* The conditions for perfect reconstruction in this case are obtained in a manner similar to that described in [8]. They are cyclic counterparts of the corresponding linear relationships and are presented below. The power complementarity condition which $H_0(e^{j2\pi k/N})$ needs to satisfy is

$$| H_0(e^{j2\pi k/N}) |^2 + | H_0(-e^{j2\pi k/N}) |^2 = 2 \qquad (14)$$

$k = 0, 1, \ldots, N-1$. For perfect reconstruction we choose

$$H_1(e^{j2\pi k/N}) = -e^{-j2\pi(N-1)k/N} H_0(-e^{-j2\pi k/N}) \qquad (15)$$

This gives

$$F_0(e^{j2\pi k/N}) = e^{-j2\pi(N-1)k/N} H_0(e^{-j2\pi k/N}) \qquad (16)$$

$$F_1(e^{j2\pi k/N}) = e^{-j2\pi(N-1)k/N} H_1(e^{-j2\pi k/N}) \qquad (17)$$

Equations (14), (15), (16) and (17) ensure that the filters satisfy the equivalent of the paraunitary conditions in this domain [8, 3] and hence satisfy the cyclic orthogonality relationships given by

$$\sum_{n=0}^{N-1} h_i(n) h_j((n+2\ell) \bmod N) = \delta(i,j)\delta(2\ell \bmod N, 0) \qquad (18)$$

where $\ell \in \mathcal{Z}$ and $i, j = 0, 1$. Note that these are cyclic equivalents of similar relationships satisfied by orthogonal or paraunitary filter banks [8].

4.5 Filter Design

To obtain the filters $H_1(e^{j2\pi k/N})$, $F_0(e^{j2\pi k/N})$ and $F_1(e^{j2\pi k/N})$ we first need to design the filter $H_0(e^{j2\pi k/N})$. For this we require the half band filter $H(e^{j2\pi k/N})$ defined as

$$H(e^{j2\pi k/N}) = H_0(e^{j2\pi k/N}) H_0(e^{-j2\pi k/N}) \qquad (19)$$

$k = 0, 1, \ldots, N-1$. Note that $H(e^{j2\pi k/N})$ is a zero phase filter. Furthermore from (14), it is clear that it has the characteristics of a *half band filter*[8, 3]. Thus, design of $H_1(e^{j2\pi k/N})$ or $H(k)$ is equivalent to assigning a value to each DFT coefficient as follows. Assuming $0 \le H(k) \le 1$, for some $H(k)$

1. $H(N-k) = H(k)$
2. $H(N/2-k) = 1 - H(k)$
3. $H(N/2+k) = H(N/2-k)$

$H_0(k)$ can now be designed by taking into account the fact that

$$H(e^{j2\pi k/N}) = | H_0(e^{j2\pi k/N}) |^2 \qquad (20)$$

Therefore

$$| H_0(e^{j2\pi k/N}) | = H(e^{j2\pi k/N})^{1/2} \qquad (21)$$

Given that in general, $H_0(e^{j2\pi k/N})$ has the form

$$H_0(e^{j2\pi k/N}) = | H_0(e^{j2\pi k/N}) | \, e^{j\phi(k)} \qquad (22)$$

we can now add the phase term $\phi(t)$. Since we require that $h_0(n)$ be real, $\phi(k)$ is antisymmetric about $N/2$. The filters $H_1(k)$, $F_0(k)$ and $F_1(k)$ can now be derived using the relations (15), (16) and (17).

Example. $N = 8$.
Let $H(0) = 0.75$. Then $H(4) = 0.25$. Let $H(1) = 0.37$. Then $H(7) = 0.37$ and $H(3) = H(5) = 0.63$. Finally, $H(2) = 1 - H(2) = H(6) = 0.5$. Thus,
$H(k) = \{0.75, 0.37, 0.5, 0.63, 0.25, 0.63, 0.5, 0.37\}$, and hence
$h(n) = \{0.5, 0.0166, 0.0, 0.1084, 0.0, 0.1084, 0.0, 0.0166\}$
Note that the non-zero even indexed points of $h(n)$ have value 0.
From (21) this gives us
$| H_0(k) | = \{0.866, 0.6082, 0.7071, 0.7937, 0.5, 0.7937, 0.7071, 0.6082\}$

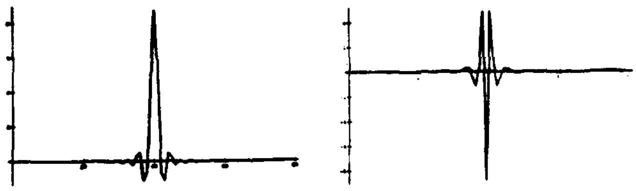


Figure 3. Low pass and high pass filter impulse responses obtained by frequency sampling the power spectrum of the Meyer scaling function.

Let us choose
$\phi(k) = \{0, 1.1168, 0.2302, -2.6746, 0.0, 2.6746, -0.2302, -1.1168\}$
This completes the design of of $H_0(k)$. The filters $H_1(k)$, $F_0(k)$ and $F_1(k)$ can now be determined. Note the flexibility and ease of design that this method offers for designing filters to specification.

5 FREQUENCY SAMPLED IMPLEMENTATION OF THE MEYER SCALING FUNCTION AND WAVELET

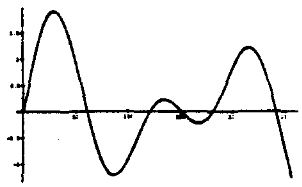
From Section 2 we observe that the power spectrum of the Meyer scaling function $\mid \Phi(\omega) \mid^2$, satisfies the conditions of being an half band filter. It follows that if it is properly sampled then the half band properties will be retained over the discrete set of samples thus obtained. To determine the rate at which it needs to be sampled, it is important to note that the function needs to be sampled symmetrically about the angular frequency of π units. This means, if the required filter size is N samples, where N is assumed even, then the samples should be $\Delta\omega = 4\pi/N$ units apart in frequency. As explained in subsection 4.5, the square root of these samples gives the samples of the low pass filter which can now be used in the PRCC framework. The shape of the low pass and the high pass filters thus obtained are shown in figure 3.

6 SYMMETRIC EXTENSION

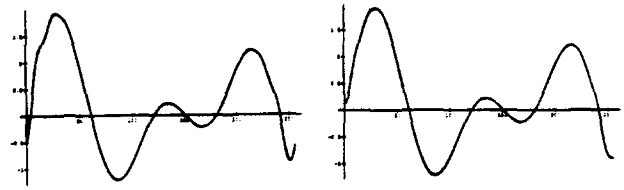
Because of circular convolution inherent in the PRCC framework, a multiresolution representation of a signal that differs considerably at its ends will suffer from distortion at its edges. In order to minimize this distortion, we symmetrically extend the block of the signal. This is done by reflecting the signal about the $(N-1)^{th}$ sample and discarding the last sample. This gives a signal length of $2N-2$. Note that the filters used in this case are also of length $2N-2$. This type of extension considerably reduces edge effects as shown in figure 4. In addition, the length of the signal subjected to decomposition at each level is always even. This allows us to decompose a signal down to a larger number of levels. However, note that this improvement is achieved at the expense of greater computational complexity.

7 CONCLUSION

In this paper, we introduced the idea of PRCC filter banks and demonstrated how it could be used for an invertible frequency sampled implementation of bandlimited DWT. Our work is different from work proposed by several authors on fast implementation of FIR filter banks in terms of the FFT algorithm [6]. It is more than an implementation. It is a novel framework for perfect reconstruction based in discrete frequency that can be applied to problems such as the frequency sampled implementation of bandlimited wavelet transforms.



(a)



(b) (c)

Figure 4. Reconstructed low pass component after 3 levels of decomposition. (a) Original signal, (b) PRCC filter banks, (c) PRCC filter bank with signal symmetrically extended.

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