AFFINE STATIONARY PROCESSES WITH APPLICATIONS TO FRACTIONAL BROWNIAN MOTION*

Birsen Yazıcı
General Electric Company
Corporate Research and Development Center
1 Reserach Circle, Niskayuna NY 12309 USA

Rangasami L. Kashyap
Purdue University
School of Electrical Engineering
W. Lafayette, IN 47906 USA

ABSTRACT

In our previous work, we introduced a new class of nonstationary stochastic processes whose spectral representation is associated with the wavelet transforms and established a mathematical framework for the analysis of such processes [1]. We refer to these processes as *affine stationary* processes. These processes are indexed by the affine group, or $ax+b$ group, which can be though of as a group of shifts and scalings. Affine stationary processes are nonstationary in the classical sense. However, their second order statistical properties are invariant under the affine group composition law.

In this paper, we show that any physically realizable affine stationary process is a wavelet transform of the white noise process. As a result, we derive a spectral decomposition of the affine stationary processes using wavelet transform. Additionally, we apply our results to the fractional Brownian motion (fBm). We show that fBm is an affine stationary process and the filter associated with the fBm is a continuous time analyzing wavelet. Finally, we apply our results to choose an optimal wavelet filter in the development of a spectral representation of fBm via wavelet transforms.

I. INTRODUCTION

Our development of theory of affine stationary processes is motivated by three facts: First, affine stationary processes are natural candidates for modeling random signals observed at multiple scales because, within the same scale an affine stationary process reduces to an ordinary stationary process, and across the scales to a class of self-similar processes which we refer to as *scale* stationary processes [2]-[4]. Second, we observe that a broad range of physical processes of practical interest are affine stationary. In particular, wavelet transform of the white noise process and self-similar processes with stationary increments are affine stationary. Third, and most importantly, we show that affine stationary processes are precisely the class of stochastic processes whose spectral decomposition is associated with the wavelet transforms.

In this paper, we give several illustrative examples of affine stationary processes. We show that any physically realizable affine stationary process can be represented as a wavelet transform of the white noise process. In particular, we derive the analyzing wavelet associated with the fractional Brownian motion. Finally, we present an orthonormal decomposition of the affine stationary processes via wavelet transforms and apply our results to develop a spectral decomposition of fractional Brownian motion in the context of affine stationary processes.

The rest of the paper is organized as follows. In Section II, we review the affine stationary processes and define some of the basic concepts. In Section III, we give examples of affine stationary processes. In Section IV, we introduce the spectral representation of affine stationary processes and discuss the analyzing wavelet associated with the fractional Brownian motion. Finally, in Section V, we discuss briefly further items of interest in this context and conclude the paper.

II. AFFINE STATIONARY PROCESSES

Definition 4.1: A stochastic process $\{X(a,b), a>0, -\infty<b<\infty\}$ is called affine stationary if it satisfies the following conditions:

i) $E[X(a,b)] = const. \quad a>0, -\infty<b<\infty.$

ii) $E[|X(a,b)|^2] < \infty \quad a>0, -\infty<b<\infty$

iii) $E[X(a_1,b_1)\overline{X(a_2,b_2)}] = E[X(\lambda a_1, \lambda b_1+\tau)\overline{X(\lambda a_2, \lambda b_2+\tau)}]$

or

$E[X(a_1,b_1)\overline{X(a_2,b_2)}] = E[X(\lambda a_1, \tau a_1+b_1)\overline{X(\lambda a_2, \tau a_2+b_2)}]$

for all $a_1, a_2, \lambda > 0$,. and $-\infty < b_1, b_2, \tau < \infty$ □

Depending on whether the process satisfies first or second condition in *iii)* or both we call it *left*, *right* and *two way* affine stationary. We refer to a and b as scale and shift indices, respectively. From Definition 2.1, it immediately follows that

$$E[X(a_1,b_1)X(a_2,b_2)] = R_l\left(\frac{a_1}{a_2}, \frac{1}{a_2}(b_1-b_2)\right) \tag{2.1a}$$

and

$$E[X(a_1,b_1)X(a_2,b_2)] = R_r\left(\frac{a_1}{a_2}, b_1 - b_2\frac{a_1}{a_2}\right). \tag{2.1b}$$

We refer to R_l and R_r as left and right affine autocorrelation functions, respectively. Within the same scale, i.e., $a_1=a_2=a$, a left affine stationary process reduces to an ordinary wide sense stationary process with the following affine autocorrelation function:

$$E[X(a,b_1)X(a,b_2)] = R_l\left(1, \frac{1}{a}(b_1-b_2)\right). \tag{2.2a}$$

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Hence, a left affine stationary process can be viewed as an assemble of dilated ordinary wide sense stationary process. On the other hand, across the scales for a fixed shift index, i.e., $b_1 = b_2 = b$, the process reduces to a class of self-similar processes known as *scale stationary* process [2]-[4]. In this case, the affine autocorrelation function is given by

$$E[X(a_1,b)X(a_2,b)] = R_l\left(\frac{a_1}{a_2},0\right). \quad (2.2b)$$

Likewise, a right affine stationary process is an ordinary wide sense stationary process. However, unlike the left affine stationary process, it exhibits the same ordinary wide sense stationary behavior at all scales. To keep our discussion simple, we shall restrict ourselves to the analysis of left affine stationary processes. We now give some examples of affine stationary processes.

III. EXAMPLES

Example 3.1: Linear Affine Stationary Processes: Informally speaking, ordinary linear processes are filtered white noise processes whereby the filtering is the ordinary convolution operation. In analogy to ordinary linear processes, we propose a linear process via affine group convolution operation. Consider the following process :

$$X(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}\Psi\left(\frac{t-b}{a}\right)dB(t), \quad a>0 \quad -\infty<b<\infty \quad (3.1a)$$

where $B(t)$, $-\infty<t<\infty$ is the Brownian motion process, and $\Psi \in L^2(R,dt)$. We can show by direct calculation that the process $\{X(a,b), a>0, -\infty<b<\infty\}$ is left affine stationary with the affine autocorrelation function

$$E[X(a,b)X(1,0)] = R(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}\Psi(t)\Psi\left(\frac{t-b}{a}\right)dt, \quad (3.1b)$$

$a>0$, $-\infty<b<\infty$. If the filter Ψ satisfies the admissibility condition of the continuous wavelet transform [5], i.e., $\int\left\{\left|\hat{\Psi}(\omega)\right|^2 / |\omega|\right\}d\omega<\infty$ where $\hat{\Psi}$ is the Fourier transform of Ψ, then $\{X(a,b), a>0, -\infty<b<\infty\}$ can be viewed as the wavelet transform of the continuous time white noise process. In addition, the admissibility condition assures that the affine autocorrelation function is square summable. This, in turn, implies that the process is physically realizable. □

Example 3.2: In this example, we will show that the wavelet transform of the *fractional Brownian motion* (fBm) is an affine stationary process up to a multiplicative factor. The fractional Brownian motion, $\{B_H(t), -\infty<t<\infty\}$, is a second order nonstationary random process with parameter H, $0<H<1$. It reduces to the standard Brownian motion for $H=1/2$. Formally, it is defined as follows :

$$B_H(t) - B_H(0) = 1/\Gamma(H+1/2)$$
$$\left\{\int_{-\infty}^{0}\left\{|t-v|^{H-1/2} - |v|^{H-1/2}\right\}dB(v) + \int_{0}^{t}|t-v|^{H-1/2}dB(v)\right\}$$

and $B_H(0)=0$ with prob. 1, $-\infty<t<\infty$ (3.2a)

Its correlation function is given by [6]

$$C(t_1,t_2) = \frac{\sigma^2}{2}\left[|t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H}\right],$$

where $\sigma^2 = \dfrac{\Gamma(1-2H)\cos(\pi H)}{\pi H}$. (3.2b)

Now, consider the following process:

$$X(a,b) = a^H\int_{-\infty}^{\infty}\Psi(t)B_H(at+b)dt, \quad a>0, -\infty<b<\infty. \quad (3.3)$$

where Ψ is a summable, zero mean analyzing wavelet , i.e., $\int|\Psi(t)|dt<\infty$ and $\int\Psi(t)dt=0$. By utilizing the self-similarity property of fBm, and using the correlation function given in (3.2b) and the assumptions on the analyzing wavelet, we obtain

$$E[X(a_1,b_1)X(a_2,b_2)] =$$
$$-\sigma^2\left(\frac{a_2}{a_1}\right)^H\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(u)\Psi(v)\left|u-\frac{a_2}{a_1}v-\frac{1}{a_1}(b_2-b_1)\right|^{2H}dudv \quad (3.4)$$

Hence, $\{X(a,b), a>0, -\infty<b<\infty\}$ is a left affine stationary process with affine autocorrelation function given as in (3.4). □

Example 3.3: Let $\{S_H(t), -\infty<t<\infty\}$ be a finite variance, self similar process with stationary increments and let H be its self similarity parameter. Consider the following increment process:

$$X(s,t) = S_H(t+s) - S_H(t) \quad s>0, -\infty<t<\infty. \quad (3.5)$$

For fixed s, the process $s>0, -\infty<t<\infty$ is an ordinary wide sense stationary process. On the other hand, for fixed t, it is self similar with parameter H. Indeed, it is left affine stationary up to an amplitude factor because

$$X(\lambda s, \lambda t+\tau) = S_H(\lambda t+\lambda s+\tau) - S_H(\lambda t+\tau)$$
$$\equiv \{S_H(\lambda t+\lambda s) - S_H(\lambda t)\} \equiv \lambda^{-H}\{S_H(t+s) - S_H(t)\} = \lambda^{-H}X(s,t) \quad (3.6)$$

where $\equiv$ denotes the equality in the sense of finite joint distributions. As a particular example, consider the increments of the fractional Brownian motion introduced in Example 3.2 :

$$A(s,t) = s^{-H}\{B_H(s+t) - B_H(t)\}, \quad s>0, -\infty<t<\infty. \quad (3.7a)$$

It is straightforward to show that the left affine autocorrelation function of the process, $\{A(s,t), s>0, -\infty<t<\infty\}$, is given by

$$E[A(s,t)A(s\lambda, s\tau+t)] =$$
$$\sigma^2\lambda^{-H}\left\{|\tau+\lambda|^{2H} + |\tau-1|^{2H} - |\tau+\lambda-1|^{2H} - |\tau|^{2H}\right\} \quad (3.7b)$$

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Hence, affine stationarity provides a mathematical framework to analyze self-similar processes with stationary increments. □

IV. SPECTRAL DECOMPOSITION OF AFFINE STATIONARY PROCESSES AND WAVELET ANALYSIS OF FRACTIONAL BROWNIAN MOTION

The spectral decomposition of ordinary wide sense stationary processes is achieved by Fourier transform. In analogy, the spectral decomposition of affine stationary processes is achieved by the generalized Fourier transform of the affine group, namely the wavelet transform. In this section, we shall develop a version of the spectral decomposition theorem for affine stationary processes and discuss briefly its practical implications in the analysis of fractional Brownian motion. We shall motivate the spectral decomposition of affine stationary processes via the development of spectral decomposition of ordinary stationary processes. First, we state the following theorem.

Theorem 4.1 : For any affine stationary process, $\{X(a,b), a>0, -\infty<b<\infty\}$, there is a filter, $f\in L^2(R,dt)$, such that the process can be represented as a linear affine stationary process in the mean square sense, i.e.,

$$X(a,b)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f\left(\frac{t-b}{a}\right)dB(t). \quad \square \tag{4.1}$$

Proof 4.1 : See [7] and [8].

We shall call $f\in L^2(R,dt)$ the *linear affine filter* associated with X. Note that a similar observation can be made for the ordinary wide sense stationary processes, in which case the filtering is defined via the additive group convolution or the ordinary convolution operation. For any wide sense stationary process, $X(t)$, $-\infty<t<\infty$, there is a filter, $h\in L^2(R,dt)$ such that X can be represented as $X(t)=\int h(t-\tau)dB(\tau)$ in the mean square sense. Moreover, under some regularity conditions, one can show that the power spectral density function, $S(\omega)$, of $X(t)$, $-\infty<t<\infty$ is given by $S(\omega)=|H(\omega)|^2$ where $H(\omega)$ is the Fourier transform of the filter h. The Parseval's equality also assures that $\|R\|=\|S\|$, where R is the autocorrelation function of the process.

As an example to illustrate Theorem 4.1, consider the increment process $\{A(s,t), s>0, -\infty<t<\infty\}$, associated with the fractional Brownian motion introduced in Example 3.3. Using Eq. (3.2a), one can show that

$$A(s,t)=\frac{1}{\Gamma(H+1/2)\sqrt{s}} \left\{\int_{-\infty}^{t}\left|1-\left(\frac{v-t}{s}\right)\right|^{H-1/2}dB(v)-\int_{-\infty}^{t}\left|\frac{v-t}{s}\right|^{H-1/2}dB(v)\right\}, \tag{4.2a}$$

$s>0$ and $-\infty<t<\infty$.

Let

$$f(v)=\frac{1}{\Gamma(H+1/2)}\left\{|1-v|^{H-1/2}u(1-v)-|v|^{H-1/2}u(-v)\right\}, \tag{4.2b}$$

$-\infty<v<\infty$ where u is the unit step function.

One can show easily that $\|f\|=\sigma^2$, where σ^2 is the variance of fBm. Hence f is the linear affine filter associated with the Fractional Brownian motion. Figure 1 and 2 show the behavior of the filter f for various parameter values of H. Note that for Brownian motion, the affine is a square wave with unit amplitude. The tail of the affine filter becomes heavier as the self-similarity parameter $H\to 1$. This is consistent with the fact that the long term correlation of the fractional Brownian motion is stronger as $H\to 1$, because the moving average or the affine filter assigns larger weight to the past observations as the parameter $H\to 1$. On the other hand, for $0<H<1/2$, the energy of the filter concentrates mostly between 0 and 1 indicating the short term correlation of the fractional Brownian motion for $0<H<1/2$. We now state the following important observation:

Theorem 4.2 : For $1/2<H<1$, the linear affine filter f associated with the fractional Brownian motion process is a continuous time analyzing wavelet. □

Proof 4.2 : See [7] and [8].

Intuitively speaking, Theorem 4.1 and 4.2 imply that inverse wavelet filtering of the fBm process, with analyzing wavelet being the linear affine filter, whitens the fBm process. This observation is rigorously stated in the following spectral decomposition theorem. However, before we introduce this theorem, let us recall the spectral representation theorem of the ordinary wide sense stationary processes. A wide sense stationary process can be viewed as linear combinations of sinusoids whose amplitudes form a nonstationary white noise process with variance equal to the spectral density function of the process, i.e., $x(t)=\sum_{n\in Z} e^{-i2\pi nt}c_n w_n$, where $\{w_n\}$ is a white noise process with unit variance and $c_n=|H(2\pi n)|=\sqrt{S(2\pi n)}$. Similarly, we shall decompose an affine stationary process into a linear combination of orthonormal functions with amplitudes forming a non-stationary white noise process whose variance satisfies a Parseval type of relationship.

Theorem 4.3 : Let $\{X(a,b), a>0, -\infty<b<\infty\}$, be an affine stationary process and $f\in L^2(R,dt)$ be the associate linear affine filter. Then for a given orthonormal time-scale wavelet basis, $\{\Psi_{n,m}, n,m\in Z\}$, of $L^2(R,dt)$, the process has the following representation in the mean square sense:

$$X(a,b)=\sum_{n,m\in Z}\tilde{\Psi}_{n,m}(a,b)c_{n,m}w_{n,m} \tag{4.3a}$$

where $\tilde{\Psi}_{n,m}(a,b)=\left\langle\sqrt{a}f(a\cdot+b),\Psi_{n,m}\right\rangle/\left\langle f,\Psi_{n,m}\right\rangle$,

$c_{n,m}=\left\langle f,\Psi_{n,m}\right\rangle$, and $\{w_{n,m}, n,m\in Z\}$ is a zero mean, unit variance white noise process. Moreover, if the affine

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autocorrelation function R of the process is square summable, the decomposition satisfies the following Parseval's relationship in the mean square sense:

$$\frac{1}{C_f}\int_0^{\infty}\int_{-\infty}^{\infty}|R(\lambda,\tau)|^2\frac{d\lambda}{\lambda^2}d\tau=\sum_{n,m\in Z}|c_{n,m}|^2 \quad (4.3b)$$

where C_f is a constant given by the Fourier transform $\hat{f}$ of the filter f, namely, $C_f=\int|\hat{f}(\omega)|^2/|\omega|d\omega$. □

Proof 4.3 : See [7] and [8].

One can easily show that $\{\tilde{\Psi}_{n,m}, n,m\in Z\}$ forms an orthonormal basis for the wavelet transform domain [7]-[8]. We shall refer the variance $|c_{n,m}|^2$ of the non-stationary white noise process, $\{c_{n,m}w_{n,m}\}$, as the spectral density function of the affine stationary processes. Theorem 4.3 implies that any physically realizable affine stationary process is a wavelet transform of the white noise process. The key issue in the analysis of affine stationary signals is the choice of the analyzing wavelet in order to simplify the spectral decomposition. In the case of fBm process, simplification of the representation means representing the process by a linear combination of orthonormal functions and white noise process in which only a small number of coefficients is non-zero. Intuitively speaking, the best analyzing wavelet is the linear affine filter associated with the process because, it leads to a single non-zero coefficient in the spectral domain. However, in practice since the parameter H is unknown, such an approach is not applicable to the processing of real data. Instead, depending on the signal processing task, one may need to choose a suboptimal wavelet filter among the filters described in (4.2b).

V. CONCLUSION

In this paper, we reviewed a new class of nonstationary signals for the analysis of self-similar and multiresolution signals. We developed basic concepts, such as autocorrelation function and shift and scale indices. We presented the utility of the proposed class by several examples of practical interest, and developed a mathematical framework for the analysis of the proposed class. In particular, we showed that any physically realizable affine stationary process is a wavelet transform of the white noise process. Also, we developed an orthonormal decomposition for affine stationary processes using wavelet transforms. Additionally, we showed that fractional Brownian motion can be viewed as an affine stationary process. We derived the wavelet filter associated with the fractional Brownian motion. We showed that inverse wavelet filtering whitens the fractional Brownian motion and comment on how one can choose the appropriate wavelet filter for a given signal processing task.

We shall report the application of affine stationary processes in wide band and mutiresolution statistical signal processing in our future work.

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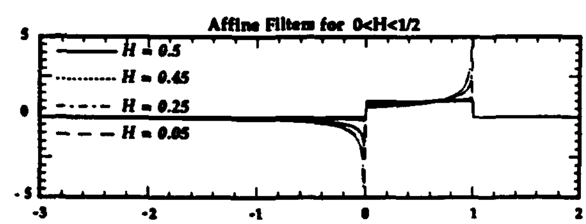
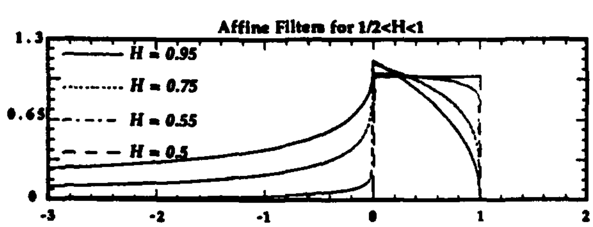
Figure1



Continuos time wavelet filters associated with the fractional Brownian motion for $0 < H < 1/2$.

Figure 2



Continuos-time wavelet filters associated with the fractional Brownian motion for $1/2 < H < 1$.

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