NON-STATIONARY WIENER FILTERING BASED ON EVOLUTIONARY SPECTRAL THEORY

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ABSTRACT

In this paper we consider solutions to the non-stationary Wiener filtering problem using the evolutionary spectral theory. Two cases of interest result from the uncorrelation between the desired signal and the noise. One constrains the support of the generating kernels of the signals and the other imposes orthogonality on their innovation processes. The latter condition is more general and our solution coincides with the one presented previously by Abdrabbo and Priestley. For the first case, we develop a new solution that depends directly on the Wold-Cramer models of the desired and noisy processes. Implementation is achieved in both cases by estimating the kernels for the Wold-Cramer representations from the spectra using the evolutionary maximum entropy spectral estimation. An example illustrating the filtering is given.

1. INTRODUCTION

The non-stationary Wiener filtering problem [1, 2] consists in obtaining a causal, linear and time-varying estimator for a desired signal $x(n)$ embedded in noise $\eta(n)$. This can be done by minimizing a mean square error between the desired signal and its estimate. Data are the present and past values of the observed signal

$$y(n) = x(n) + \eta(n). \tag{1}$$

In this paper we will consider the solution of the Wiener filtering problem using the evolutionary spectral theory [1]. Other methods [3] have also been proposed to solve this problem.

It will be shown that using the Wold-Cramer representation [4] of the signals involved, and the orthogonality principle, the above non-stationary Wiener filtering problem can be solved. Two cases of interest result from the uncorrelation between $x(n)$ and $\eta(n)$ which can be obtained by either constraining the support of the kernels that generate $x(n)$ and $\eta(n)$, or by imposing the orthogonality of their innovation processes. The latter condition is more general and our solution coincides with that of Abdrabbo and Priestley [5]. The first case is analogous to the stationary case when the Wiener filter is a band-pass filter with a frequency bandwidth coinciding with that of the spectrum of the desired signal. Although conceptually similar, our solution provides an LTV filter capable of separating the desired signal from the noise.

Implementation of the solutions requires estimation of the kernels for the Wold-Cramer representations of $x(n)$ and $y(n)$ from their spectra. Such estimation is possible by means of the evolutionary maximum entropy [6].

2. NON-STATIONARY WIENER FILTERING

Consider the observed process in equation (1), and assume $\{x(n)\}$ and $\{\eta(n)\}$ are uncorrelated, zero-mean non-stationary processes with evolutionary spectral densities $S_x(n,\omega)$ and $S_\eta(n,\omega)$. We are interested in finding an estimator of $x(n+m)$ (for some $m \geq 0$) of the form,

$$\widehat{x}(n+m) = \sum_{u=0}^{\infty} b(n,u)y(n-u) \tag{2}$$

and such that it minimizes the mean-square error (MSE)

$$\varepsilon_n(m) = E|x(n+m) - \widehat{x}(n+m)|^2. \tag{3}$$

Remarks

(1) As in the stationary case, when $m > 0$ the above problem is prediction and when $m = 0$ it is filtering.

(2) The Wold-Cramer representations [4] for $x(n)$ and $\eta(n)$ are:

$$\begin{aligned} x(n) &= \int_{-\pi}^{\pi} H_x(n,\omega)e^{j\omega n}dZ_x(\omega) \\ \eta(n) &= \int_{-\pi}^{\pi} H_\eta(n,\omega)e^{j\omega n}dZ_\eta(\omega) \end{aligned}$$

which can be viewed as the outputs of LTV systems with innovation processes of the form

$$e(n) = \int_{-\pi}^{\pi} e^{j\omega n}dZ(\omega) \tag{4}$$

where $Z(\omega)$ is an orthogonal increment process. $H(n,\omega)$ is the generalized transfer function of a linear time-varying (LTV) system defined as,

$$H(n,\omega) = \sum_{m=0}^{\infty} h(n,m)e^{-j\omega m}. \tag{5}$$

(3) For $x(n)$ and $\eta(n)$ to be uncorrelated we need that

$$E[x(n)\eta^*(n)] = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H_x(n,\omega_1)H_\eta^*(n,\omega_2)e^{j(\omega_1-\omega_2)n} E[dZ_x(\omega_1)dZ_\eta^*(\omega_2)] \tag{6}$$

be equal to zero. This can be attained assuming that $Z_x(\omega)$ and $Z_\eta(\omega)$ are mutually orthogonal, i.e., $E[dZ_x(\omega_1)dZ_\eta^*(\omega_2)] = 0$ for all ω_1 and ω_2. On the other hand, if $Z_x(\omega) = Z_\eta(\omega) = Z(\omega)$ and $Z(\omega)$ is an orthogonal increment process we have that equation (6) becomes

$$E[x(n)\eta^*(n)] = \int_{-\pi}^{\pi} H_x(n,\omega)H_\eta^*(n,\omega)d\omega \quad (7)$$

and uncorrelation then requires that the above integral be equal to zero. A condition which is satisfied, for instance, when the spectra $S_x(n,\omega)$ and $S_\eta(n,\omega)$ do not overlap.
(4) If $Z_x(\omega)$ and $Z_\eta(\omega)$ are mutually uncorrelated, we then have

$$E[dZ_x^*(\omega_1)dZ_y(\omega_2)] = \frac{H_x(n,\omega_1)}{H_y(n,\omega_1)}d\omega_1, \quad (8)$$

as can be easily verified from $E[x(n)y^*(n)] = E[|x(n)|^2]$.

3. MINIMIZATION OF MSE

The minimization of $\varepsilon_n(m)$ can be done using the orthogonality principle, according to which $\widehat{x}(n+m)$ is the projection of $x(n+m)$ onto a plane spanned by the data $\{y(k), k \leq n\}$. Thus the estimation error is orthogonal to the data or to a linear combination of it, $\widehat{x}(n+m)$, i.e.,

$$E[x(n+m) - \widehat{x}(n+m)]\widehat{x}^*(n+m) = 0. \quad (9)$$

Replacing the Wold-Cramer representation of $y(n-u)$ in equation (2) we get that

$$\begin{aligned}\widehat{x}(n+m) &= \sum_{u=0}^{\infty} b(n,u)\int_{-\pi}^{\pi} H_y(n-u,\omega)e^{j\omega(n-u)}dZ_y(\omega) \\ &= \int_{-\pi}^{\pi} G(n,\omega)e^{j\omega n}dZ_y(\omega), \quad (10)\end{aligned}$$

where

$$\begin{aligned}G(n,\omega) &= \sum_{u=0}^{\infty} b(n,u)H_y(n-u,\omega)e^{-j\omega u} \\ &= \sum_{v=0}^{\infty}\sum_{u=0}^{v} b(n,u)h_y(n-u,v-u)e^{-j\omega v} \quad (11)\end{aligned}$$

where we have replaced $H_y(n-u,\omega)$ and used the fact that $h_y(n-u, v-u) = 0, v-u < 0$, due to causality. Using equations (8) and (10) we first have that

$$E[x(n+m)\widehat{x}^*(n+m)] = \int_{-\pi}^{\pi} \frac{S_x(n+m,\omega)}{H_y^*(n+m,\omega)}e^{j\omega m} G^*(n,\omega)d\omega \quad (12)$$

and then that

$$E[\widehat{x}(n+m)\widehat{x}^*(n+m)] = \int_{-\pi}^{\pi} |G(n,\omega)|^2 d\omega. \quad (13)$$

Equation (9) becomes

$$\int_{-\pi}^{\pi} \left[\frac{S_x(n+m,\omega)}{H_y^*(n+m,\omega)}e^{j\omega m} - G(n,\omega)\right] G^*(n,\omega)d\omega = 0. \quad (14)$$

Remarks
If the uncorrelation of $x(n)$ and $\eta(n)$ is due to equation (7) being zero, while $Z_x(\omega) = Z_\eta(\omega) = Z(\omega)$, we have that

$$E[x(n+m)\widehat{x}^*(n+m)] = \int_{-\pi}^{\pi} H_x(n+m,\omega)e^{j\omega m} G^*(n,\omega)d\omega \quad (15)$$

and equation (13) still holds so that equation (9) becomes

$$\int_{-\pi}^{\pi} \left[H_x(n+m,\omega)e^{j\omega m} - G(n,\omega)\right] G^*(n,\omega)d\omega = 0. \quad (16)$$

3.1. Normal Equations

Suppose then that

$$\frac{S_x(n+m,\omega)}{H_y^*(n+m,\omega)}e^{j\omega m} = F^{(1)}(n,\omega) + F^{(2)}(n,\omega) \quad (17)$$

where $F^{(1)}(n,\omega)$ is a backward and $F^{(2)}(n,\omega)$ is a forward polynomial in $e^{j\omega}$ with time–varying coefficients. Equation (14) can then be expressed as

$$\int_{-\pi}^{\pi} F^{(2)}(n,\omega)G^*(n,\omega)d\omega + \int_{-\pi}^{\pi} \left[F^{(1)}(n,\omega) - G(n,\omega)\right] G^*(n,\omega)d\omega = 0. \quad (18)$$

The first integral can be shown to be zero and the second can be made zero by letting

$$G(n,\omega) = F^{(1)}(n,\omega). \quad (19)$$

Letting $F^{(1)}(n,\omega)$ be of the form

$$F^{(1)}(n,\omega) = \sum_{v=0}^{\infty} l(n+m, v+m)e^{-j\omega v} \quad (20)$$

then from equation (19) we obtain the final expression for the normal equations of the non-stationary Wiener filter,

$$\sum_{u=0}^{v} b(n,u)h_y(n-u,v-u) = l(n+m,v+m), \qquad v \geq 0 \quad (21)$$

which coincide with the solution obtained by Abdrabbo and Priestley in [5].

Remarks
(1) In a similar manner, if the uncorrelation of $x(n)$ and $\eta(n)$ is due to equation (7) being zero, we can write

$$H_x(n+m,\omega)e^{j\omega m} = \widetilde{F}^{(1)}(n,\omega) + \widetilde{F}^{(2)}(n,\omega) \quad (22)$$

and our minimization condition can easily be shown to be,

$$G(n,\omega) = \widetilde{F}^{(1)}(n,\omega) \quad (23)$$

where

$$\widetilde{F}^{(1)}(n,\omega) = \sum_{v=0}^{\infty} h_x(n+m, v+m)e^{-j\omega v} \quad (24)$$

We then get the following normal equations for this case:

$$\sum_{u=0}^{v} b(n,u)h_y(n-u,v-u) = h_x(n+m,v+m), v \geq 0. \quad (25)$$

(2) The minimum mean square error can easily be shown to be

$$\epsilon_n^{min}(m) = \int_{-\pi}^{\pi} \frac{S_x(n+m,\omega)S_\eta(n+m,\omega)}{S_y(n+m,\omega)} d\omega + \int_{-\pi}^{\pi} |F^{(2)}(n,\omega)|^2 d\omega \quad (26)$$

for the general case with normal equations (21). For the case with normal equations (25) we get

$$\tilde{\epsilon}_n^{min}(m) = \int_{-\pi}^{\pi} |\tilde{F}^{(2)}(n,\omega)|^2 d\omega. \quad (27)$$

3.2. Implementation

To implement equation (21) or (25) we need to obtain estimates of the kernels $H_x(n,\omega)$ and $H_y(n,\omega)$ from the given spectrum $S_x(n,\omega)$ and $S_y(n,\omega)$. According to (5) we can then obtain estimates for $h_x(n,u)$ and $h_y(n,u)$ to use directly in (25). For equation (21), the estimate $\widehat{H}_y(n,\omega)$ is combined with $S_x(n,\omega)$ as in (17) to obtain for each n its separation into causal and anticausal components. Estimates $\widehat{H}_x(n,\omega)$ and $\widehat{H}_y(n,\omega)$ can be obtained from the evolutionary maximum entropy spectral estimation [6].

The evolutionary maximum entropy spectral estimation consists in maximizing the entropy

$$\int_{-\pi}^{\pi} \ln S(n,\omega) d\omega \quad (28)$$

for every n, under the conditions that $S(n,\omega) > 0$, $\forall n$, and that the Fourier coefficients of $S(n,\omega)$, $\{f(n,\mu)\}$, are matched for $0 \leq \mu \leq P_n$. This becomes the classical maximum entropy problem for each n, yielding

$$\widehat{H}(n,\omega) = \frac{\sqrt{\epsilon(n)}}{A(n,\omega)} \quad (29)$$

where,

$$A(n,\omega) = [1 + \sum_{k=1}^{P} a(n,k)e^{-j\omega k}]. \quad (30)$$

The $\{a(n,k)\}$ coefficients and $\epsilon(n)$ can be obtained from the Levinson's algorithm

$$\sum_{k=0}^{P_n} a(n,k)f(n,k-\mu) = -f(n,\mu) \qquad 0 \leq \mu \leq P_n \quad (31)$$

Thus, given $S_x(n,\omega)$ and $S_y(n,\omega)$ we find the corresponding Fourier coefficients $\{f_x(n,\mu), f_y(n,\mu)\}$ and obtain for each n $\widehat{H}_x(n,\omega)$ and $\widehat{H}_y(n,\omega)$.

4. EXAMPLE

We present a filtering example ($m = 0$) using the new normal equations in (25), and we assume that the spectra of $x(n)$ and $\eta(n)$ overlap in only a small set of time–frequency points. As a measure of performance, we use the following SNR improvement index,

$$SNR = 10\log_{10} \frac{\sum_n |\eta(n)|^2}{\sum_n |x(n) - \widehat{x}(n)|^2}. \quad (32)$$

Our noisy observation signal $y(n)$ consists of a desired sinusoidal frequency modulated signal $x(n)$ corrupted by stationary white noise $\eta(n)$ (0 dB SNR). The evolutionary spectrum of $y(n)$ is shown in Fig. 1. The evolutionary spectra of the desired and recovered signals are shown in Figs. 2 and 3, respectively. The method used to estimate the spectra was the evolutionary maximum entropy [6]. The SNR improvement achieved in this case is of 3.8 dB. Figure 4 displays the Wiener filter's frequency response showing the adaptation along the FM sinusoidal chirp.

5. CONCLUSIONS

We have shown that the non-stationary Wiener filter can be formulated and solved using the evolutionary spectral theory. Although less general, a case of special interest is when the uncorrelation between the desired and the noise signal is due to a special condition on the corresponding Wold–Cramer kernels. Its solution is an LTV filter capable of separating the desired signal and the noise. If $x(n)$ and $\eta(n)$ are stationary, the solutions given here coincide with the ones in that case [1, 2].

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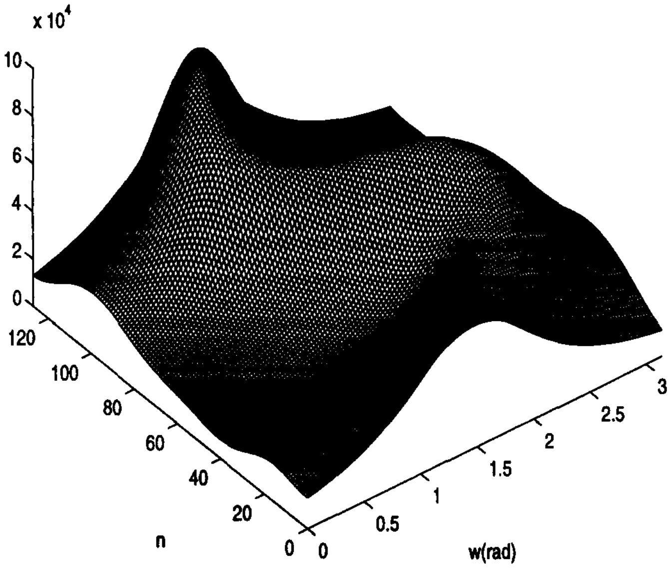

Figure 1. $S_y(n, \omega)$

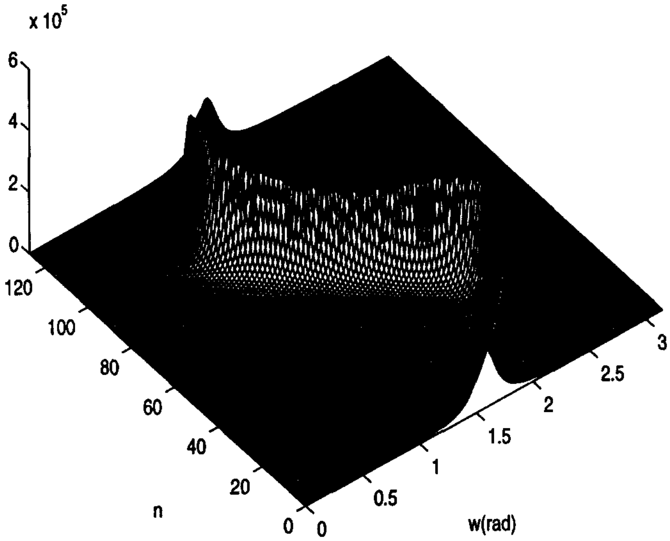

Figure 2. $S_x(n, \omega)$

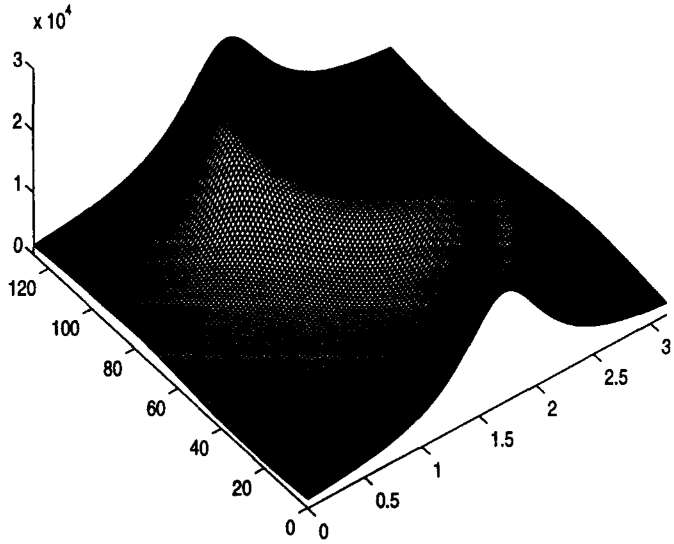

Figure 3. $S_{\hat{x}}(n, \omega)$

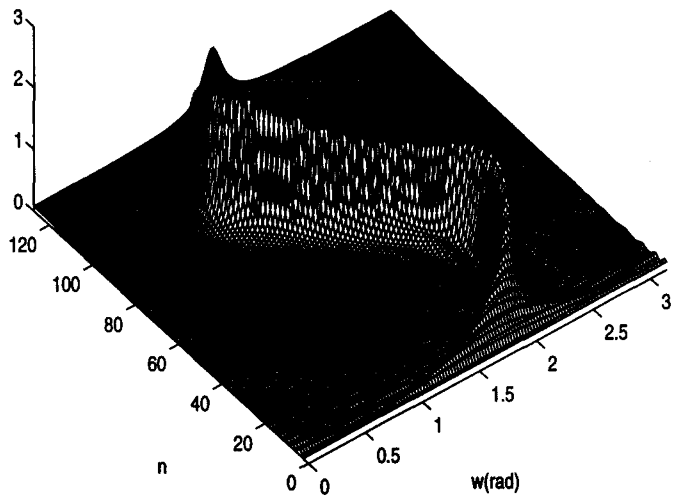

Figure 4. $|B(n, \omega)|^2$ of the Wiener filter.