GENERALISED AUGMENTATION APPROACH FOR ARBITRARY LINEAR ANTENNA ARRAYS

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ABSTRACT

We investigate DOA (direction-of-arrival) estimation for arbitrary linear arrays, where the antenna positions may be non-integer values in half-wavelength units. We introduce an approach based on arbitrary virtual linear arrays to resolve manifold ambiguity and estimate DOA's in the superior case. These virtual arrays adopt the set of covariance lags specified by the original array and so themselves have an incomplete set of covariance lags. A maximum entropy completion algorithm for the partially-specified Hermitian covariance matrix is proposed. This is followed by an algorithm which searches for a fixed number of plane wavefronts ("generalised Pisarenko completion"). The variety of possible virtual array geometries also permits a "randomised" approach, whereby the DOA estimates are determined as the stable point of partial solutions calculated over the set of particular virtual geometries. Numerical simulations demonstrate the high efficiency of manifold ambiguity resolution, and a remarkable proximity to the Cramer-Rao bound for DOA estimation.

1. INTRODUCTION

Most current research in the field of DOA estimation for nonuniform linear arrays (NLA's) focuses on one specific class of array, broadly known as "minimum-redundancy" arrays [1, 2]. For such M-element arrays, the sensor positions d_i $(i = 1, \dots, M)$ can only take integer values, usually measured in half-wavelength $(\lambda/2)$ units.

Let us consider the class of arbitrary-geometry NLA's, for which the sensor positions may be non-integer values (once again measured in half-wavelength units). Classical examples of ambiguous geometries belong to this class [3]. Obviously the non-integer inter-element spacing makes the standard augmentation approach [2] unsuitable for arbitrary-geometry arrays.

Thus for the *superior case* $m \geq M$ the only existing techniques that may be applied are the multidimensional maximum likelihood search [4] and the geometry-invariant model-fitting approach [5]. Even for the *conventional case* $m < M$ when standard MUSIC-type algorithms can be used, the problem of manifold ambiguity resolution is sufficient motivation to search for a new ambiguity-free approach, analogous to augmentation for integer-geometry NLA's [6].

The necessity to resolve manifold ambiguities for the conventional case and the need to find suitable tools for dealing with the superior case has stimulated the emergence of the generalised augmentation approach introduced below.

Throughout this paper, we measure spatial frequency w in units of $2d\sin\theta/\lambda$, so that $w \in [-1, 1]$.

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2. RESULTS

We will use the four-element arbitrary-geometry NLA discussed by Proukakis and Manikas [3]

$$d = \{0, 1.2, 3.4, 4.6\} \tag{1}$$

as our illustrative example. They demonstrated that if this antenna operates in the presence of three sources with DOA's

$$w_3 = \{0.1304, 0.5652, 0.7451\}, \tag{2}$$

then the MUSIC algorithm will provide five directions rather than three (see Fig. 1(a), dotted line). However, the Cramer-Rao bound in this case is finite for any finite N. For example, using the input signal model with unknown powers, $N = 1000$ snapshots and 20dB SNR, the $2m$-variate Fisher information matrix gives us $\mathrm{CRB}(w_3) = 0.0007$. The theoretical existence of useful DOA estimates for this manifold ambiguity situation encourages us to search for a new approach to resolve manifold ambiguity.

We begin with the observation that in the limit $N \to \infty$, the sample direct data covariance (DDC) matrix $\hat{R}$ provides the asymptotically-optimal estimates for the following covariance lags

$$r_\kappa = \{r_0, r_{1.2}, r_{2.2}, r_{3.4}, r_{4.6}\}. \tag{3}$$

Thus we can construct an arbitrary virtual array with inter-element spacings corresponding to precisely these covariance lags. For example, we may introduce the array

$$d' = \{0, 1.2, 2.2, 3.4, 4.6\} \tag{4}$$

which has one additional "virtual" element. For this virtual array the DOA set w_3 is no longer manifoldly-ambiguous, and the MUSIC pseudo-spectrum provides the three true directions (see Fig. 1(a), dashed line).

Obviously the difference set r_κ is incomplete for this virtual array, as is the 5-variate augmented covariance matrix:

$$H = \begin{bmatrix} r_0 & r_{1.2} & r_{2.2} & r_{3.4} & r_{4.6} \\ r^*_{1.2} & r_0 & ? & r_{2.2} & r_{3.4} \\ r^*_{2.2} & ? & r_0 & r_{1.2} & ? \\ r^*_{3.4} & r^*_{2.2} & r^*_{1.2} & r_0 & r_{1.2} \\ r^*_{4.6} & r^*_{3.4} & ? & r^*_{1.2} & r_0 \end{bmatrix} \tag{5}$$

which may be completed using the maximum-entropy (ME) criterion. This completion method is similar to the ME completion method proposed for partially-augmentable integer arrays [7]. The ME Hermitian completion H_{ME} can be uniquely defined with respect to the optimality condition defined by the theorem in [8] using convex programming techniques [9], since feasibility is guaranteed for deterministic covariance lags. In [8] it was proven that the inverse of the ME-completed matrix has a zero entry in every location corresponding to an unspecified entry in the matrix H.

Fig. 1(b) shows the ME spectrum for the ME-completed matrix H_{ME} (dotted line) compared with that of the true covariance matrix H_{exact} (dashed line) corresponding to the array geometry d' and the source scenario w_3. One can see that ME completion in this particular example does not immediately resolve the ambiguity, since both the ME spectrum and the MUSIC pseudo-spectrum for H_{ME} (Fig. 1(c), dashed line) are still far from the true ones. The main reason for this is that ME completion for arbitrary geometries does not necessarily retain a plane-wave structure in the restored wavefronts.

The following *generalised Pisarenko approach* is proposed for true plane-wave completion:

Step 1: For the given incomplete covariance matrix H and virtual geometry d, define the unique ME completion H_{ME} by the above-mentioned convex programming technique.

Step 2: Find the p.d. Toeplitz matrix corresponding to some uniform linear array (ULA) with the ME spectrum closest in the least-squares sense to the ME spectrum of the Hermitian matrix H_{ME}. Note that for a good approximation the number of elements in this virtual ULA should exceed the number of virtual nonuniform antenna elements. Moreover, since the ME spectrum of H_{ME} is no longer a periodic function on $w \in [-1, 1]$, the inter-element spacing for the virtual ULA should be chosen to be less than the standard half-wavelength. The details of ME-equalisation may be found in [10], wherein the unique solution which relies on the Gohberg-Semencul [11] formula is derived.

Step 3: For this p.d. Toeplitz covariance matrix T corresponding to the virtual ULA, find the fixed signal subspace approximation T_{GS} by alternating projections [10]. Note that peaks in the MUSIC spectrum of T_{GS} may arise in the area of imaginary DOA's ($|\sin\theta| > 1$); these should be eliminated using orthogonal projections (a constrained MUSIC technique).

Step 4: Use the signal powers p_i and DOA's θ_i obtained from the matrix T_{GS} as initial estimates in an iterative refinement procedure. We derive the linear expansion for the array manifold in the neighbourhood of these estimates to solve the LMSE fitting problem for the specified covariance lags r_κ.

Fig. 1(c) (dotted line) illustrates the MUSIC pseudo-spectrum of T_{GS} for our example, while the final refined DOA's obtained by LMSE fitting are absolutely accurate.

This generalised Pisarenko approach may also be used for DOA estimation in the superior case with $m = 4$ sources. Fig. 1(d) illustrates the ME spectrum of the ME-completed matrix H_{ME} (dashed line) and the MUSIC pseudo-spectrum of the Toeplitz approximation T_{GS} (dotted line) for the source scenario

$$w_4 = \{-0.23, 0.04, 0.66, 0.87\}. \qquad (6)$$

Once again, the final solution obtained after DOA-fitting is absolutely accurate.

In these simulations, we have used $d/\lambda = 0.4$ for the virtual 10-element ULA. It should be noted that this may lead to incorrect DOA estimation for endfire situations ($\sin\theta = \pm 1$). Moreover, the alternating projections may occasionally lead to a set of DOA's which are sufficiently far from the true ones to prevent successful LMSE fitting.

If we now turn our attention to a stochastic sample covariance matrix rather than the deterministic one, the feasibility condition for the existence of the ME completion is no longer guaranteed [8]. A simple minimal diagonal loading is proposed to meet this condition, defined by the Ellipsoid Algorithm [9].

Table 1 shows the bias and RMS error for each DOA in a stochastic simulation of 1000 trials involving the source scenario w_4. The sample volume is $N = 1000$ snapshots and the SNR is 20dB. The above generalised Pisarenko approach achieves very good accuracy compared with the corresponding Cramer-Rao bounds.

true DOA (w)	-0.23	0.04	0.66	0.87
bias	0.0053	0.0052	0.0062	0.0065
RMSE	0.0041	0.0042	0.0048	0.0051
total error	0.0068	0.0067	0.0079	0.0083
CRB	0.0065	0.0060	0.0073	0.0079

Table 1. Sample stochastic simulation results.

Note that the geometry d' is far from being the only way to construct a virtual array. Treating the estimated lags in Eqn. (3) as the consecutive inter-element lags

$$r_{i,i+1} = r_\kappa, \qquad \kappa = 2, \ldots, 5 \qquad (7)$$

we are free to create sets of geometries with an arbitrary number of elements, with the inter-element spacings drawn from the set of admissible separations. For example, a 5-element virtual array might be defined as

$$d'' = \{0, 1.2, 3.4, 6.8, 11.4\}. \qquad (8)$$

By this approach, the three principal diagonals of the augmented matrix H are always specified, while the remaining specified entries have arbitrary positions in the matrix. This matrix may once again be completed by convex programming, and the above approach may be applied to the augmented matrix H_{ME} to obtain the DOA estimates.

A less accurate but computationally more efficient approach can be introduced by simply ignoring all off-tridiagonal specified elements. By the above-mentioned theorem, the inverse of the ME completion (H_{ME}^{-1}) is tridiagonal. In fact, the tridiagonal matrix H_3 is the simplest member of a class of banded matrices with the single condition $r_0^2 - |r_\kappa|^2 > 0$ being necessary and sufficient for the existence of a p.d. completion, including the ME completion with a tridiagonal inverse [12]. Because of this property, the ME completion can be defined analytically as described in [13], and the MUSIC algorithm may be applied to the augmented matrix H_{ME} to obtain the partial DOA estimates.

Of course, each particular virtual array geometry is manifoldly ambiguous, and such a simplified approach leads to significant errors in each set of partial DOA estimates. The main idea is to treat the set of all possible geometries as a random set. The different geometries produce *different* manifold ambiguities and *different* partial DOA errors, and hence we expect the true DOA estimates to be the stable points of this random set. The true DOA estimates can now be properly inferred from the partial (sample) MUSIC pseudo-spectra.

The simplest approach for such a combination pseudo-spectrum $f_\Sigma(\theta)$ is a scaled product. This method has been analysed for the above ambiguity resolution example involving w_3. The array geometry in this case was restricted to $M_\alpha = 1 + \frac{1}{2}M(M-1) = 7$ elements, and all (redundant) lags of the original array have been used to create the random set. One hundred trials were performed for each value of L, and deterministic covariance lags were used. The three absolute maxima of the function $\log f_\Sigma(\theta)$ associated with over the ambiguous set of five DOA's

$$w_5 = \{-1, -0.4492, 0.1304, 0.5652, 0.7451\}. \qquad (9)$$

are indeed coincident with the true values with a probability of 0.91 for only $L = 8$ random permutations, and probability 0.97 for $L = 13$. For $L > 25$, all 100 trials resulted in true DOA identification.

Number of snapshots (N)	20	100	500	1000	5000	∞
Probability of correct identification	0.478	0.714	0.819	0.869	0.912	0.912

Table 2. Example of probability convergence for DOA estimation by association.

For stochastic covariance estimation, a relatively large sample volume N is necessary to obtain a sufficiently high probability of true identification. Table 2 shows the probability of correct identification as a function of sample volume for the manifold ambiguity example w_3 with 20dB input SNR for each source. Each of 1000 trials involved both random snapshots and $L = 8$ random permutations used in the averaging process. The convergence rate is quite modest, which is to be expected due to the completion of the extremely sparse matrix $\hat{H}_3$.

Thus the proposed randomised augmentation approach demonstrates the capability of resolving manifold ambiguity for arbitrary-geometry NLA's, obviously providing that the Cramer-Rao bound is finite (*ie.* not an inherent ambiguity). It should be clear that in terms of DOA estimation accuracy this approach cannot compete with our earlier rigourous completion procedure which uses all available covariance lags.

Note that inherent ambiguity conditions are also dependent on *a priori* assumptions similarly to partially-augmentable arrays [6]. For example, for the DOA set w_5 the $2m$-variate Fisher matrix J_{2m} is rank deficient, while J_m gives the reasonable optimum accuracy $\text{CRB}(w_5) = 0.0531$ for $N = 100$ and 20dB SNR. This means that in this case when the source powers are known, the source DOA's can be properly estimated, while the absence of *a priori* power values makes the problem of DOA estimation ambiguous.

3. SUMMARY

The results obtained in this paper demonstrate that DOA estimation problems for arbitrary linear arrays may be treated by the described generalised augmentation approach, which relies upon associated virtual array geometries.

This approach leads to the problem of optimal completion for incomplete Hermitian matrices. For the maximum entropy (ME) criterion, the unique optimal completion exists for any feasible initial condition. When the specified covariance lags are precisely known (deterministic completion), the optimum solution always exists and may be found by a computationally-efficient convex programming method. For the set of sample covariance lags, the feasibility condition is not necessarily satisfied and an additional minimal diagonal loading is found to meet this condition.

Unlike our earlier investigation into partially-specified Toeplitz matrix completion [7], the ME spectrum of the ME-completed Hermitian matrix differs significantly from the true ME spectrum of the virtual array. For the more general Hermitian case, ME completion does not necessarily retain the plane-wave structure of the source field. Therefore a plane-wave completion method (the generalised Pisarenko approach) has been described. This method begins with the ME spectrum of the ME-completed Hermitian matrix, then finds the p.d. Toeplitz matrix (and corresponding virtual uniform linear array) with the most similar ME spectrum. Signal eigen-subspace truncation produces the p.d. Toeplitz matrix with a fixed number of planar signals which in most cases are located in close proximity to the true DOA's. Finally, a LMSE adjustment refines these DOA estimates to fit the specified set of covariance lags.

By this approach, ambiguity resolution has been demonstrated for manifoldly ambiguous situations; also DOA estimation for the superior case (of four sources and four elements) has resulted in estimation accuracy remarkably close to the corresponding Cramer-Rao bounds.

A "randomised" augmentation approach which exploits the fusion of the MUSIC pseudo-spectra obtained for a "random" set of virtual array geometries has also been proposed. It was shown that this method is able to provide statistically reliable ambiguity resolution with a rather modest number of snapshots, using a simplified analytic ME completion of the augmented covariance matrix truncated to tridiagonal form.

Needless to say, ambiguity resolution is only possible for inherently unambiguous situations where the Fisher matrix is not rank-deficient (*ie.* is positive definite).

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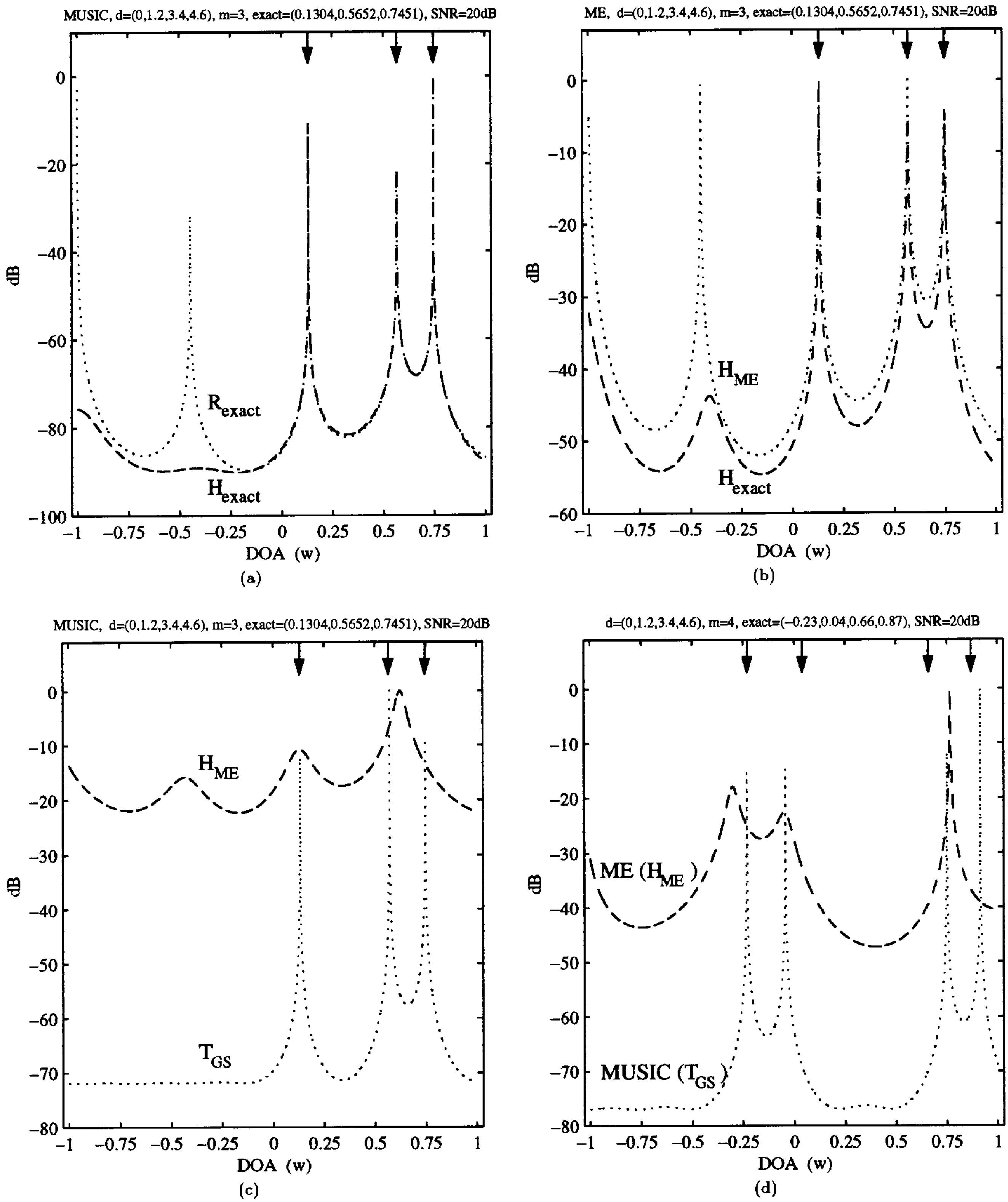

Figure 1. (a), (b) and (c) show various deterministic MUSIC and ME spectra for the three-source example; (d) compares the initial and penultimate stages of the generalised Pisarenko approach for the four-source example. Arrows mark the position of the true DOA's and the final absolutely accurate estimated DOA's.