

DIRECTION OF ARRIVAL TRACKING BELOW THE AMBIGUITY THRESHOLD

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ABSTRACT

We present an algorithm for direction-of-arrival tracking that allows operation below the ambiguity threshold of the direction finding system. Using multiple target tracking techniques, the algorithm turns the most likely directions-of-arrival of each measurement into multiple potential tracks and then selects the true track as that with the maximum cumulative likelihood. The improvement offered by the algorithm, namely the extension of the ambiguity-free domain, is demonstrated by simulated experiments.

1. INTRODUCTION

One of the parameters characterizing the performance of every direction-finding (DF) system is the ambiguity threshold. The ambiguity threshold is the signal-to-noise ratio (SNR) below which, for a given number of samples, the probability of an ambiguous direction-of-arrival (DOA) result rises and thereby sets a limit on the performance of the DF system.

In this paper we address the problem of tracking a moving source using a DF system that operates below the ambiguity threshold. This problem is of interest since none of the existing DOA tracking algorithms, [1]-[3], can handle the frequent, large and non-random error resulting from operation below the ambiguity threshold.

2. PROBLEM FORMULATION

For simplicity we formulate the problem for the case of a single source. The extension to the case of multiple sources is relatively straightforward. Suppose that a moving source emits a narrowband signal, and that the signal is received by an array consisting of p sensors. For simplicity assume that both the source and the sensors are confined to a plane and that the source is in the far-field of the array.

Using complex envelope notation, the signals re-

ceived by the array can be expressed by

$$\mathbf{x}(t) = \mathbf{a}(\theta(t))s(t) + \mathbf{n}(t) \quad (1)$$

where $\theta(t)$ is the source DOA, $s(t)$ is its signal as received at a reference point, $\mathbf{n}(t)$ is the additive noise, and $\mathbf{a}(\theta)$ is the steering vector towards direction θ .

Suppose that we estimate $\theta(t)$ every T seconds from batches of m samples taken at $\{t_i\}_{i=1}^m$ from the array output, and that the source dynamics can be modeled by its angular position $\theta(t)$ and angular velocity $\dot{\theta}(t)$ using the following discrete constant-velocity state-space model:

$$\begin{aligned} \mathbf{y}(k+1) &= \mathbf{F}\mathbf{y}(k) + \mathbf{w}(k) \\ \check{\theta}(k) &= \mathbf{H}\mathbf{y}(k) + v(k) \end{aligned} \quad (2)$$

where $\mathbf{y}(k) = [\theta(kT), \dot{\theta}(kT)]^T$ is the state vector, $\check{\theta}(k)$ is the source estimated DOA, $\mathbf{w}(k)$ and $v(k)$ are the process and measurement noise, respectively, and $\mathbf{F}$ and $\mathbf{H}$ are state transition matrix and measurement matrix, respectively, given by

$$\mathbf{F} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad \mathbf{H} = [1 \quad 0] \quad (3)$$

Our problem can now be stated as follows: Given the data batches, estimate the DOA track $\theta(t)$.

To solve the problem we assume that the following conditions hold:

- A1:** *The change in the DOA of the source during the batch time is negligible.*
- A2:** *The change in the DOA of the source between consecutive batches is small. e.g. $\theta(kT) \approx \theta((k+1)T)$.*
- A3:** *The emitted signal $s(t)$ is an unknown and arbitrary waveform.*
- A4:** *The additive noise samples $\{\mathbf{n}(t_i)\}$ are i.i.d Gaussian complex random vectors with zero mean and covariance $\zeta^2 \mathbf{I}$, where ζ^2 is unknown.*

A5: The process noise $\mathbf{w}(k)$ is Gaussian distributed with zero mean and covariance matrix

$$\mathbf{Q} = r \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix}, \quad (4)$$

where r is a known constant representing the acceleration variance [4].

A6: The measurement noise $v(k)$ is Gaussian distributed with zero mean and unknown variance $\sigma^2(k, \theta)$.

A1 is needed to insure an unbiased DOA measurement. **A2** is needed to insure proper tracking initialization. **A3** and **A4** are not critical and are included to simplify the optimal DOA estimator. **A5** is common in modeling of dynamic systems [4]. **A6** is reasonable, recalling that the measurement noise is actually the DOA estimation error.

3. SOFT-DECISION TRACKING

The common approach to DOA tracking is based on using the most likely DOA at each measurement point [1]-[3]. This approach can be described as “hard-decision”. In contrast to this approach, our approach is based on regarding *all* the high peaks of the DOA likelihood function, i.e., all the potential DOA's, as potential track points. Using data association and multiple target tracking techniques we turn these points into multiple potential tracks and then select the track with the highest cumulative likelihood at a given time kT as the DOA track for that time instance. Obviously, this approach can be described as “soft-decision”. We next present a detailed description of the different steps of this approach.

3.1. Potential Track Points Estimation

The first step of the algorithm is the estimation of the most likely directions-of-arrival from each batch. From (1), assuming that each batch consists of m samples, the condense log-likelihood function is given by the following expression [5]

$$L(\theta) = \frac{\mathbf{a}^H(\theta) \hat{\mathbf{R}} \mathbf{a}(\theta)}{\mathbf{a}^H(\theta) \mathbf{a}(\theta)}, \quad (5)$$

where $\hat{\mathbf{R}}$ is the sample-covariance matrix of the batch

$$\hat{\mathbf{R}} = \frac{1}{m} \sum_{i=1}^m \mathbf{x}(t_i) \mathbf{x}^H(t_i). \quad (6)$$

The peaks of $L(\theta)$ represent the most likely directions-of-arrival for the given batch. Above the ambiguity

threshold the peak corresponding to the DOA is the highest. Yet, below the ambiguity threshold the height of the ambiguous peaks rise and occasionally exceed the height of the peak corresponding to the DOA. In light of the structure of $L(\theta)$ below the ambiguity threshold, one approach for potential track points estimation is to select *all* the peaks of $L(\theta)$ as potential track points, i.e. the set of points obeying

$$\Theta = \{\theta : \dot{L}(\theta) = 0, \ddot{L}(\theta) < 0\}. \quad (7)$$

Alternatively, to reduce the computational load, one can limit the number of peaks to only those which are Δ dB below the value of the global maximum. Δ is a design parameter determined from the ambiguity structure and the field of view.

3.2. Track Formation

Track life cycle evolves in four stages: Initialization, Confirmation, Updating and Termination. A track is initialized whenever a potential track point is not associated with any existing track. After initialization, the track is considered as a tentative track. Until termination, for every new set of potential track points, the tentative and confirmed tracks are updated by the data association algorithm, to be described later, which associates the track points to the tracks. A tentative track is confirmed when I points are associated to it out of J consecutive attempts. The values of I and J are design parameters.

A track is terminated if it is not updated during K consecutive attempts. The value of K is a design parameter. While an active track is updated, its past values are used to predict, via a Kalman filter, the track position at the next measurement point. This predicted position is then used in the data association and in filtering the DOA track.

The Kalman filter equations are given by [4]:

$$\begin{aligned} \hat{\mathbf{y}}(k|k) &= \hat{\mathbf{y}}(k|k-1) + \mathbf{K}(k)[\check{\theta}(k) - \mathbf{H}\hat{\mathbf{y}}(k|k-1)] \\ \mathbf{K}(k) &= \mathbf{P}(k|k-1)\mathbf{H}^T \\ &\quad [\mathbf{H}\mathbf{P}(k|k-1)\mathbf{H}^T + \sigma^2(k, \theta)]^{-1} \\ \mathbf{P}(k|k) &= [\mathbf{I} - \mathbf{K}(k)\mathbf{H}]\mathbf{P}(k|k-1) \\ \hat{\mathbf{y}}(k+1|k) &= \mathbf{F}\hat{\mathbf{y}}(k|k) \\ \mathbf{P}(k+1|k) &= \mathbf{F}\mathbf{P}(k|k)\mathbf{F}^T + \mathbf{Q}. \end{aligned}$$

To carry out these recursions we must estimate the measurement noise variance $\sigma^2(k, \theta)$. To this end, note that if the number of samples per batch, m , is large enough, $\sigma^2(k, \theta)$ can be approximated by the Cramér-Rao bound since it is a relatively tight bound for these

ambiguity-free tracks. We denote the estimated measurement variance as $\hat{\sigma}^2(k, \theta)$. To initialize the Kalman filter we use the first two track points, $\tilde{\theta}(1), \tilde{\theta}(2)$, as described in [4].

3.3. Data Association

Data association amounts to the association of potential track points with the active tracks. Association is needed because the location and the height of the likelihood peaks can change substantially between consecutive DOA estimates below the ambiguity threshold.

This data association problem is simpler than the data association problem in conventional multitarget tracking since in this problem: (i) Tracks do not cross, (ii) Track separation is governed by the array beamwidth. To describe the association algorithm, let $\{\tilde{\theta}_j(k)\}_{j=1}^{N(k)}$ denote the newly generated set of potential track points at time kT , each with its corresponding likelihood value $\{L(\tilde{\theta}_j(k))\}_{j=1}^{N(k)}$. Also, let $\{T_i(k)\}_{i=1}^{M(k)}$ denote the set of active tracks at time kT , each with its cumulative likelihood value $\{L_i(k)\}_{i=1}^{M(k)}$, given by

$$L_i(k) = L_i(k-1) + L(\tilde{\theta}_i(k)), \quad (8)$$

its predicted state vector $\{y_i(k|k-1)\}_{i=1}^{M(k)}$, given by

$$\hat{y}_i(k|k-1) = [\hat{\theta}_i(k|k-1), \hat{\theta}_i(k|k-1)]^T, \quad (9)$$

and its measurement prediction variance $\{s_i(k)\}_{i=1}^{M(k)}$, given by

$$s_i(k) = E[e^2(k)] = \mathbf{H}\mathbf{P}_i(k|k-1)\mathbf{H}^T + \hat{\sigma}^2(k, \hat{\theta}_i(k|k-1)), \quad (10)$$

where $E[\cdot]$ denotes statistical expectation and

$$e(k) = \tilde{\theta}(k) - \mathbf{H}\hat{y}_i(k|k-1). \quad (11)$$

Our solution to the association problem is based on probabilistic modeling of the association process and on using the maximum a-posteriori criterion as a selection rule.

Let $p_i(k|T_i(k))$ denote the a-priori probability that the i -th track be updated by a new track point at time kT . Regarding the cumulative likelihood score $L_i(k)$ as representing this a-priori probability, a natural estimator for $p_i(k|T_i(k))$ is given by

$$\hat{p}_i(k|T_i(k)) = \frac{L_i(k)}{\sum_{n=1}^{M(k)} L_n(k)}. \quad (12)$$

Let $d_{ij}(k)$ denote the angular distance between the point $\tilde{\theta}_j(k)$ to the i -th track predicted position, $\hat{\theta}_i(k|k-1)$, and let $\mathcal{A}(k)$ denote the correct association between the existing tracks $\{T_i(k)\}_{i=1}^{M(k)}$ and the

points $\{\tilde{\theta}_j(k)\}_{j=1}^{N(k)}$. Conditioned on the correct association, $\{d_{ij}(k)\}$ can be modeled as independent and zero mean Gaussian random variables with variance $s_i(k)$. That is, the probability density of $\{d_{ij}(k)\}$ is given by

$$p(\{d_{ij}(k)\}|\mathcal{A}(k), \{T_i(k)\}_{i=1}^{M(k)}) = \prod_{(i,j) \in \mathcal{A}(k)} \frac{1}{\sqrt{2\pi s_i(k)}} \exp\left(-\frac{d_{ij}^2(k)}{2s_i(k)}\right). \quad (13)$$

Now, using Bayes' rule, the probability of $\mathcal{A}(k)$ is given by

$$p(\mathcal{A}(k)|\{d_{ij}(k)\}, \{T_i(k)\}_{i=1}^{M(k)}) = \prod_{(i,j) \in \mathcal{A}(k)} \frac{1}{\sqrt{2\pi s_i(k)}} \exp\left(-\frac{d_{ij}^2(k)}{2s_i(k)}\right) p_i(k|T_i(k)). \quad (14)$$

According to the maximum a-posteriori criterion, the most probable association is the one that maximizes (14) over all potential associations, i.e.,

$$\max_{\{\mathcal{A}_l(k)\}_{l=1}^{P(k)}} p(\mathcal{A}_l(k)|\{d_{ij}(k)\}, \{T_i(k)\}_{i=1}^{M(k)}), \quad (15)$$

where $\{\mathcal{A}_l(k)\}_{l=1}^{P(k)}$ denotes the set of all potential associations. Here, $P(K)$ denotes the cardinality of the set and is given by $P(k) = \frac{q(k)!}{(q(k)-p(k))!}$, with $q(k) = \max(M(k), N(k))$, $p(k) = \min(M(k), N(k))$.

This amounts, after taking the logarithm, to the minimization of the following criterion:

$$\min_{\{\mathcal{A}_l(k)\}_{l=1}^{P(k)}} \sum_{(i,j) \in \mathcal{A}_l(k)} \mathbf{D}_{ij}(k), \quad (16)$$

where

$$\mathbf{D}_{ij}(k) = \frac{d_{ij}^2(k)}{s_i(k)} + \log(s_i(k)) - 2\log(\hat{p}_i(k|T_i(k))). \quad (17)$$

Let $\{(i_n^*, j_n^*)\}_{n=1}^{P(k)}$ denote the set of point-track pairs that minimize (16) and let their corresponding distances be denoted by $\{D_{i_n^* j_n^*}(k)\}_{n=1}^{P(k)}$.

If $D_{i_n^* j_n^*}(k) \leq \gamma$, where γ is a threshold design parameter, the j_n^* -th input point is associated to the i_n^* -th track, and its likelihood score, $L(\tilde{\theta}_{j_n^*}(k))$, is added to the track score. When $D_{i_n^* j_n^*}(k) > \gamma$ the input point is not likely to belong to a track, and thus is used to initiate a new track.

The association procedure described above is applicable when the prediction covariance, $s_i(k)$, is available. Yet, newly initialized tracks, i.e. tracks with only one point, lack this information. In order to associate a second point to an initiated track, we compute its association distance by $\mathbf{D}_{ij}(k) = d_{ij}^2(k)$ and carry out the association using the array beamwidth as the threshold parameter γ .

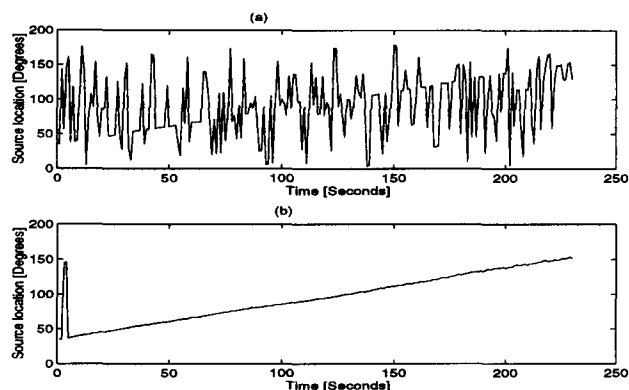


Figure 1: Track resulting for the (a) conventional method and (b) our algorithm using a three-element linear array with inter-elements spacing ratio of 3:4 and aperture of 8λ . Track length is 250 seconds, with DOA estimation done every 1 second, using 100 samples of the array output. SNR is -6dB.

3.4. Most Likely Track Selection

After the completion of the data association step, the final step at every time instance is the selection of the most likely track. To this end we use the cumulative likelihood score and select the track for which this score is maximized, i.e.

$$\hat{i}(k) = \arg \max_i (L_i(k)), \quad i = 1, \dots, M(k). \quad (18)$$

Given the most likely track, $\hat{i}(k)$, the most likely DOA at time kT is given by the filtered position of the best track, $\hat{\theta}_{\hat{i}(k)}(k|k)$.

4. SIMULATION RESULTS

To demonstrate the performance of the proposed soft-decision algorithm, we compared it with the hard-decision algorithm using the deterministic maximum likelihood (DML) estimator.

Figure 1 presents a typical track estimation result for a three-element linear array with inter-elements spacing ratio of 3:4 and 8λ aperture. The SNR is -6dB. Note the ambiguity-rich track obtained in the conventional method and the ambiguity-free track obtained in our method. The only ambiguity errors in our method occurred on the third and the fourth seconds, which are within an initialization period.

Figure 2 shows the ambiguity percentage results as a function of the SNR for $m = 100$ snapshots per batch and array apertures of 2λ , 4λ , 8λ . Note that while in the conventional approach the probability of ambiguity rises with the aperture size and is high even at very high SNR, in our approach the probability of ambiguity is

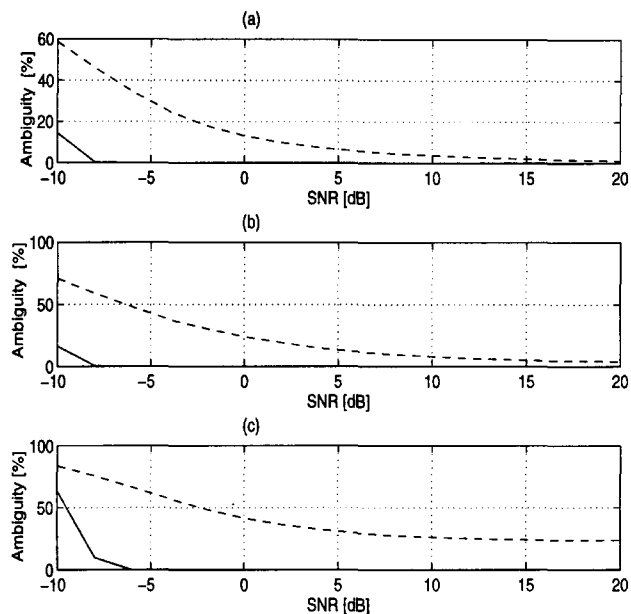


Figure 2: Ambiguity percentage versus SNR for a three-element linear array with inter-elements spacing ratio of 3:4 and aperture of (a) 2λ , (b) 4λ (c) 8λ , and for 100 samples of the array output, using the DML algorithm (dashed) and our soft-decision algorithm (solid).

essentially independent of the aperture and starts rising only at -8dB, which is the no-information threshold for this problem.

5. REFERENCES

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