

OPTIMAL PARAMETRIZATION OF TRUNCATED GENERALIZED LAGUERRE SERIES

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ABSTRACT

In this paper we address the problem of approximating functions on a semi-infinite interval by truncated series of orthonormal generalized Laguerre functions. The generalized Laguerre functions contain two parameters, namely a scale factor and an order of generalization. The rate of convergence of a generalized Laguerre series depends on the choice of these parameters. Results concerning the determination of the two parameters are presented.

1. INTRODUCTION

It is sometimes desirable to have a compact description of a function, say $f(t)$, defined on the continuous interval $\mathbb{R}^+$, which exhibits an exponential decay towards infinity. Such a function could for example be the impulse response of a causal and stable system. A good approach is to approximate the function by a truncated Laguerre series, because the Laguerre functions also show an exponential decay. The rate of decay of the Laguerre functions is determined by a free design parameter. In the case of a truncated Laguerre series the choice of this parameter is essential to obtain a small Integrated Squared Error (ISE),

Some systems have impulse responses that show a slow start and an exponential decay towards infinity. A Laguerre series expansion of such an impulse response cannot be expected to show a fast convergence, since the basis functions have an abrupt start. A much better basis in these cases would be the set of generalized Laguerre functions. Like the Laguerre functions, the generalized Laguerre functions contain a free design parameter that determines the rate of decay towards infinity. In addition, they contain a parameter that determines how fast the functions start. Finding good values for these parameters for generalized Laguerre approximations is the subject of study in this paper.

The approach in this paper was inspired by similar work on truncated Laguerre expansions in [1]-[6], and truncated Hermite expansions in [7]. The structure of the paper is as follows. First, in Section 2 we introduce the generalized Laguerre functions and some of their properties. Second, Section 3 deals with truncated expansions using generalized Laguerre functions and the stationarity conditions of the ISE with respect to the scale factor. Lastly, in Section 4 an approach is given to arrive at a "good" scale factor and a "good" order of generalization for the generalized Laguerre

functions based solely on some specific measurements of the function $f(t)$.

2. GENERALIZED LAGUERRE FUNCTIONS

The *generalized Laguerre polynomials* belong to the class of classical orthogonal polynomials, and are given by [8, 9]:

$$l_m^{(\alpha)}(x) = \sum_{n=0}^m (-1)^n \binom{m+\alpha}{m-n} \frac{x^n}{n!}, \quad (1)$$

with $m = 0, 1, \dots$; $\alpha > -1$, and $x \in \mathbb{R}^+$. The parameter α is called the order of generalization. In the special case where $\alpha = 0$ the polynomials in (1) reduce to the *Laguerre polynomials* [8, 9].

The generalized Laguerre polynomials are orthogonal under the window function $w^{(\alpha)}(x) = x^\alpha e^{-x}$, in particular

$$\int_0^\infty w^{(\alpha)}(x) l_m^{(\alpha)}(x) l_n^{(\alpha)}(x) dx = \delta_{mn} \frac{\Gamma(m+\alpha+1)}{m!},$$

where δ_{mn} is the Kronecker delta and $\Gamma(\cdot)$ is the Gamma function.

The generalized Laguerre polynomials and the weight function $w^{(\alpha)}(x)$ can be used to construct functions on $\mathbb{R}^+$ that are orthonormal with respect to a unity weight function. These functions are obtained by multiplying the generalized Laguerre polynomials by the square root of their weight function, followed by a change of scale in the x -axis according to $x = \sigma t$ ($\sigma > 0$) and a normalization, see [10]. As a result we obtain the so-called *generalized Laguerre functions*, given by

$$\lambda_m^{(\alpha)}(\sigma; t) = \sqrt{\frac{\sigma m!}{\Gamma(m+\alpha+1)}} \sum_{n=0}^m (-1)^n \binom{m+\alpha}{m-n} \frac{(\sigma t)^{n+\alpha/2}}{n!} e^{-\sigma t/2}. \quad (2)$$

Here t stands for the time index ($t \geq 0$). These functions have two free parameters, namely the order of generalization α and the scale factor σ . For $\alpha = 0$ the functions in (2) reduce to the *Laguerre functions* [11]. The orthonormality of the generalized Laguerre functions is expressed by

$$\int_0^\infty \lambda_m^{(\alpha)}(\sigma; t) \lambda_n^{(\alpha)}(\sigma; t) dt = \delta_{mn}. \quad (3)$$

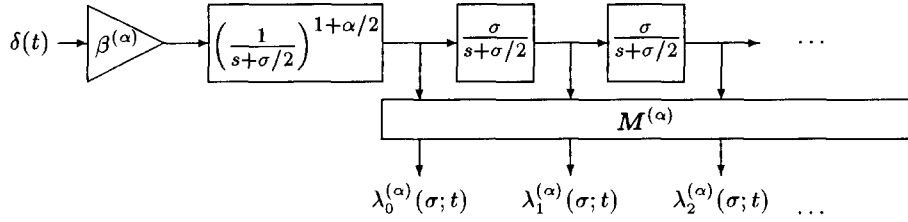


Figure 1: Network to generate the generalized Laguerre functions for even values of the order of generalization α . Further, $\beta^{(\alpha)} = \sigma^{(1+\alpha)/2}$.

In the rest of this paper an inner product as given in the left-hand side of (3) of two real functions $f(t)$ and $g(t)$ is written as $\langle f(t), g(t) \rangle$.

The generalized Laguerre functions have Laplace transforms

$$\Lambda_m^{(\alpha)}(\sigma; s) = \beta^{(\alpha)} \left(\frac{1}{s + \sigma/2} \right)^{1+\alpha/2} \sum_{n=0}^m M_{m,n}^{(\alpha)} \left(\frac{\sigma}{s + \sigma/2} \right)^n.$$

Here, $\beta^{(\alpha)} = \sigma^{(1+\alpha)/2}$ and $M_{m,n}^{(\alpha)}$ is the $(m+1, n+1)$ -th element of the lower triangular matrix $M^{(\alpha)}$, given by $M_{m,n}^{(\alpha)} = 0$ if $m < n$, and

$$M_{m,n}^{(\alpha)} = (-1)^n \binom{m}{n} \frac{\Gamma(n + \alpha/2 + 1)}{\Gamma(n + \alpha + 1)} \sqrt{\frac{\Gamma(m + \alpha + 1)}{m!}},$$

if $m \geq n$. For even values of α the Laplace transforms $\Lambda_m^{(\alpha)}(\sigma; s)$ are rational functions of s and the generalized Laguerre functions can be generated as the impulse responses of the network shown in Figure 1, see also [10].

Increasing the value of α results in a more slowly starting weight function under which the generalized Laguerre polynomials are developed. In turn, this leads to more slowly starting generalized Laguerre functions. Thus, with α , the center of energy of these functions can be shifted in time. Equivalently, a larger value for α results in more emphasis on the lower frequency components of the functions at the expense of the higher frequency components. As an example, in Fig. 2 the first four generalized Laguerre functions are plotted for two different values of α .

For later use we give that the partial derivative of the m -th generalized Laguerre function with respect to the scale factor σ can be expressed in the next and previous generalized Laguerre function [10]:

$$\frac{\partial \lambda_m^{(\alpha)}(\sigma; t)}{\partial \sigma} = D_{m,m+1}^{(\alpha)}(\sigma) \lambda_{m+1}^{(\alpha)}(\sigma; t) - D_{m-1,m}^{(\alpha)}(\sigma) \lambda_{m-1}^{(\alpha)}(\sigma; t), \quad (4)$$

with

$$D_{m-1,m}^{(\alpha)}(\sigma) = \sqrt{m(m+\alpha)/2}\sigma, \quad m = 1, 2, \dots,$$

and $D_{-1,0}^{(\alpha)}(\sigma) = 0$. For $\alpha = 0$ the relation (4) reduces to the special case of the Laguerre functions, see [2], [4] and [12]. A similar result exists for Hermite functions [7].

3. TRUNCATED GENERALIZED LAGUERRE EXPANSIONS

Let $L_2(\mathbb{R}^+)$ denote the Hilbert space of square-integrable functions on $\mathbb{R}^+$. The generalized Laguerre functions form a complete orthonormal set in $L_2(\mathbb{R}^+)$ [13]. Therefore, any square-integrable causal function $f(t)$ can be expanded in a generalized Laguerre series according to

$$f(t) = \sum_{m=0}^{\infty} c_m(\alpha, \sigma) \lambda_m^{(\alpha)}(\sigma; t), \quad (5)$$

where the expansion coefficients $c_m(\alpha, \sigma)$ are determined via the following inner product:

$$c_m(\alpha, \sigma) = \langle f(t), \lambda_m^{(\alpha)}(\sigma; t) \rangle. \quad (6)$$

An M -term truncated generalized Laguerre expansion becomes the approximation $f_M(t)$ of $f(t)$ and reads:

$$f_M(t) = \sum_{m=0}^{M-1} c_m(\alpha, \sigma) \lambda_m^{(\alpha)}(\sigma; t).$$

With the truncation error defined by $e(t) = f(t) - f_M(t)$, the ISE is given by

$$\zeta_M(\alpha, \sigma) = \langle e(t), e(t) \rangle = \sum_{m=M}^{\infty} c_m^2(\alpha, \sigma),$$

and clearly is a function of α and σ . Minimization of the ISE with respect to α and σ is a nonlinear problem. As a preliminary step towards solving this problem, it is possible to derive the stationarity conditions of the ISE with respect to the scale factor σ . By setting the derivative of $\zeta_M(\alpha, \sigma)$ with respect to σ equal to zero, one finds:

$$\frac{\partial \zeta_M(\alpha, \sigma)}{\partial \sigma} = 2 \sum_{m=0}^{M-1} c_m(\alpha, \sigma) \frac{\partial c_m(\alpha, \sigma)}{\partial \sigma} = 0. \quad (7)$$

With $c_m(\alpha, \sigma)$ given by (6) and with the specific form of the derivative of a generalized Laguerre function with respect to σ in (4) we find

$$\begin{aligned} \frac{\partial c_m(\alpha, \sigma)}{\partial \sigma} &= D_{m,m+1}^{(\alpha)}(\sigma) c_{m+1}(\alpha, \sigma) \\ &\quad - D_{m-1,m}^{(\alpha)}(\sigma) c_{m-1}(\alpha, \sigma). \end{aligned} \quad (8)$$

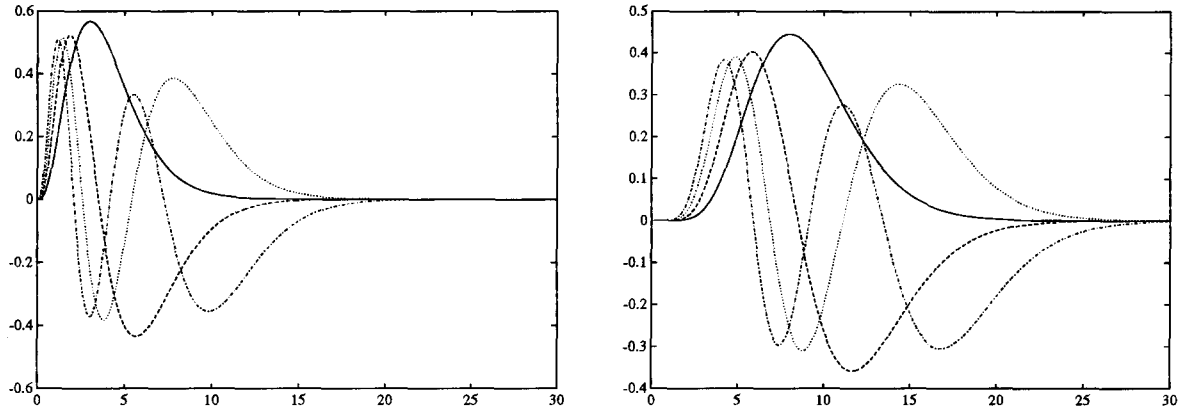


Figure 2: The first four generalized Laguerre functions with $\sigma = 2$. For the functions on the left plot we took $\alpha = 6$ and we took $\alpha = 16$ in the right plot. The solid, dashed, dotted and dash-dotted lines correspond with $m = 0$, $m = 1$, $m = 2$ and $m = 3$, respectively.

Using this, (7) becomes

$$\frac{\partial \zeta_M(\alpha, \sigma)}{\partial \sigma} = 2 \sum_{m=0}^{M-1} c_m(\alpha, \sigma) \cdot \left\{ D_{m,m+1}^{(\alpha)} c_{m+1}(\alpha, \sigma) - D_{m-1,m}^{(\alpha)} c_{m-1}(\alpha, \sigma) \right\},$$

a series of which only the final term remains uncanceled:

$$\frac{\partial \zeta_M(\alpha, \sigma)}{\partial \sigma} = \frac{\sqrt{M(M+\alpha)}}{\sigma} c_{M-1}(\alpha, \sigma) c_M(\alpha, \sigma).$$

Thus, for any stationary point of $\zeta_M(\alpha, \sigma)$ with respect to σ it is necessary and sufficient that at least one of the two following conditions is satisfied [10]:

$$c_{M-1}(\alpha, \sigma) = 0 \quad \text{or} \quad c_M(\alpha, \sigma) = 0. \quad (9)$$

The problem of finding the stationary points of $\zeta_M(\alpha, \sigma)$ with respect to σ is now reduced to finding the zeros of either $c_{M-1}(\alpha, \sigma)$ or $c_M(\alpha, \sigma)$ with respect to σ . To this end the gradient in (8) can be used. Higher-order derivatives can readily be obtained by iterative application of (8). If one has a reasonably good choice of the scale factor σ for a given order of generalization α , then a better one can be found with one or more iterations of Newton's method.

When $c_{M-1}(\alpha, \sigma) = 0$, the M -term approximation reduces to an $(M-1)$ -term approximation. Therefore, one would expect that at local minima of the ISE as a function of σ the second condition in (9) is the one satisfied. Unfortunately, this is not always the case.

In the next section it will be shown how a good choice for the scale factor and the order of generalization can be made, using only a few specific measurements of the function $f(t)$.

4. A GOOD σ AND α FOR TRUNCATED GENERALIZED LAGUERRE EXPANSIONS

Like all classical polynomials the generalized Laguerre polynomials satisfy a second-order differential equation. This

differential equation is given by [9]

$$x y_m^{(\alpha)''}(x) + (x+1) y_m^{(\alpha)'}(x) + \left(m + \frac{\alpha}{2} + 1 - \frac{\alpha^2}{4x}\right) y_m^{(\alpha)}(x) = 0, \quad (10)$$

where a prime denotes differentiation with respect to x and $y_m^{(\alpha)}(x) = e^{-x} x^{\alpha/2} l_m^{(\alpha)}(x)$. From (10) a differential equation for the generalized Laguerre functions can be found:

$$A \lambda_m^{(\alpha)''}(\sigma; t) + B \lambda_m^{(\alpha)'}(\sigma; t) + C \lambda_m^{(\alpha)}(\sigma; t) = m \lambda_m^{(\alpha)}(\sigma; t), \quad (11)$$

where a prime denotes differentiation with respect to t and

$$A = -\frac{t}{\sigma}, \quad B = -\frac{1}{\sigma}, \quad C = \frac{\sigma t}{4} - \frac{\alpha + 1}{2} + \frac{\alpha^2}{4\sigma t}.$$

As we will see hereafter, the differential equation in (11) can be used to provide a tight upper bound of the ISE that can be minimized with α and σ . Following the ideas presented in [5] we proceed as follows. We define the linear operator $\mathcal{L}$ on a function $f(t)$ in $L_2(\mathbb{R}^+)$ by

$$\mathcal{L}f(t) = A f''(t) + B f'(t) + C f(t), \quad (12)$$

for which we can write with (5) and (11)

$$\mathcal{L}f(t) = \sum_{m=0}^{\infty} m c_m(\alpha, \sigma) \lambda_m^{(\alpha)}(\sigma; t).$$

Using the linearity property of inner products we may write

$$\begin{aligned} \langle f(t), \mathcal{L}f(t) \rangle &= \sum_{m=0}^{\infty} m c_m^2(\alpha, \sigma) \\ &\geq M \sum_{m=M}^{\infty} c_m^2(\alpha, \sigma) \end{aligned}$$

which yields the following tight upper bound for the ISE:

$$\zeta_M(\alpha, \sigma) = \sum_{m=M}^{\infty} c_m^2(\alpha, \sigma) \leq \frac{F(\alpha, \sigma)}{M}, \quad (13)$$

where $F(\alpha, \sigma) = \langle f(t), \mathcal{L}f(t) \rangle$. Denoting by $\mathcal{C}_F$ the class of functions with given $F(\alpha, \sigma)$, it is easy to prove the existence of a function $g(t) \in \mathcal{C}_F$ which achieves the bound in (13), see [5].

For functions $f(t)$ in $L_2(\mathbb{R}^+)$ we have that

$$\langle f(t), tf''(t) \rangle = -\langle f(t), f'(t) \rangle - \langle f'(t), tf'(t) \rangle.$$

With (12) this gives us

$$F(\alpha, \sigma) = \frac{\alpha^2}{4\sigma} m_{-1} - \frac{\alpha+1}{2} m_0 + \frac{\sigma}{4} m_1 + \frac{1}{\sigma} m_2,$$

where the moments m_{-1} to m_2 are defined as

$$\begin{aligned} m_{-1} &= \langle f(t), t^{-1} f(t) \rangle, & m_0 &= \langle f(t), f(t) \rangle, \\ m_1 &= \langle f(t), t f(t) \rangle, & m_2 &= \langle f'(t), t f'(t) \rangle, \end{aligned}$$

assuming that these exist. Note that all the moments are strictly positive numbers.

Setting the derivative of $F(\alpha, \sigma)$ with respect to σ equal to zero yields a "good" scale factor $\hat{\sigma}$ for a given order of generalization α :

$$\hat{\sigma}(\alpha) = \left(\frac{\alpha^2 m_{-1} + 4m_2}{m_1} \right)^{\frac{1}{2}}. \quad (14)$$

Indeed, with $\alpha = 0$ the scale factor in (14) corresponds to the one found by Parks in [3] (there the notation $p = 2\sigma$ is used).

Setting the derivatives of $F(\alpha, \sigma)$ with respect to α and σ both equal to zero yields one unique solution for a "good" scale factor and a "good" order of generalization:

$$\begin{aligned} \hat{\sigma} &= 2\sqrt{\frac{m_{-1}m_2}{|m_1m_{-1} - m_0^2|}}, \\ \hat{\alpha} &= \frac{m_0}{m_{-1}} \hat{\sigma}. \end{aligned}$$

The number of generalized Laguerre functions in the truncated expansion can be chosen such that the bound in (13) is satisfactorily small. Finally, if desired, the scale factor can be refined with one or more iterations of Newton's method.

For the generalized Laguerre functions themselves (with $\alpha > 0$) it can be proved that

$$\begin{aligned} m_{-1} &= \frac{\sigma}{\alpha}, \\ m_0 &= 1, \\ m_1 &= \frac{\alpha + 2m + 1}{\sigma}, \\ m_2 &= \frac{\sigma(2m + 1)}{4}, \end{aligned}$$

so that $\hat{\sigma} = \sigma$ and $\hat{\alpha} = \alpha$, which is reassuring.

5. CONCLUSION

In this paper some results concerning the determination of the optimal center and scale factor for truncated generalized Laguerre series were presented. It was demonstrated that the stationary points of the ISE with respect to the scale

factor satisfy a simple condition, which also holds in the general case of a system identification setting with non-white excitation [10].

An easy-to-use procedure has been proposed to directly find a "good" scale factor and a "good" order of generalization. The method does not require complete knowledge of $f(t)$, only a few specific measurements of the function need to be known, and is thus useful in practical situations with experimental data. The obtained scale factor and order of generalization are the best that can be found when only the afore-mentioned measurements are available.

6. REFERENCES

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