ON MOMENTS OF COMPLEX WISHART AND COMPLEX INVERSE WISHART DISTRIBUTED MATRICES

Dirk Maiwald *Dieter Kraus*
STN ATLAS Elektronik GmbH
Simulation Division/Naval Division
Sebaldsbrücker Heerstr. 235, D–28305 Bremen

ABSTRACT

This paper addresses the calculation of moments of complex Wishart and complex inverse Wishart distributed random matrices. Complex Wishart and complex inverse Wishart distributed random matrices are used in applications like radar, sonar, or seismics in order to model the statistical properties of complex sample covariance matrices and complex inverse sample covariance matrices, respectively. Moments of these random matrices are often needed e.g. in studies of asymptotic properties of parameter estimates. This paper gives a derivation of the probability density function of complex inverse Wishart distributed random matrices. Furthermore, strategies are outlined for the calculation of the moments of complex Wishart and complex inverse Wishart distributed matrices.

1. INTRODUCTION

In signal processing the multivariate complex normal distribution is used in order to describe the statistical properties of certain signals. An example are signals measured by sensor arrays in radar, sonar, or seismic applications. Often data enters an estimation procedure in form of a sample covariance matrix $\hat{\mathbf{S}} = \frac{1}{K}\sum_{k=1}^{K} \underline{x}^k \underline{x}^{*k}$. In sonar applications, $\underline{x}^k$ denotes the vector valued finite Fourier transformed sensor outputs of succeeding data stretches $(k = 1, \ldots, K)$. Then, the vectors $\underline{x}^k$ are asymptotically normally distributed. In radar applications a quadrature demodulation scheme is used to obtain the complex envelope of the signal. The complex envelope is modelled by the complex normal distribution in radar detection problems and source location estimation.

Typically estimators like e.g. maximimum likelihood, least squares, or weighted subspace fitting estimators for source location estimation are functions of the sample covariance matrix $\hat{\mathbf{S}}$. The derivation of the asymptotic properties of these estimates when the sample size K increases involves moments of complex Wishart or complex inverse Wishart distributed matrices. This is because the sample covariance matrix is distributed as a complex random Wishart matrix if measured data is modelled by a complex normally distributed random vector. Furthermore, for the derivation of the asymptotic properties the expected value or the covariance matrix of a function of $\hat{\mathbf{S}}$ are needed. If an algorithm uses the inverse of the sample covariance matrix (e.g. [3]) one is lead to the complex inverse Wishart distribution.

In this paper, the probability density function of inverse complex Wishart distributed random matrices is derived. The moments of complex Wishart distributed matrices are determined using the characteristic function. A novel strategy for the calculation of the moments of inverse complex Wishart distributed random matrices is outlined. The complex Wishart distribution is discussed in [4], results for the real Wishart distribution can be found e.g. in [1].

2. PRELIMINARIES

Following [4] a complex N-dimensional random vector $\underline{X}$ is called complex normally distributed with expected value $\underline{\mu}$ and hermitian positiv (semi-)definite covariance matrix $\mathbf{\Sigma}$: $\underline{X} \sim \mathcal{N}_N^{\mathbb{C}}\left(\underline{\mu}, \mathbf{\Sigma}\right)$, if the $2N$-dimensional real vector $\underline{Y} = (\mathrm{Re}\,\underline{X}', \mathrm{Im}\,\underline{X}')'$ is normally distributed with expected value $\underline{\mu}_R = \left(\mathrm{Re}\,\underline{\mu}', \mathrm{Im}\,\underline{\mu}'\right)'$ and covariance matrix

$$\mathbf{\Sigma}_R = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\,\mathbf{\Sigma} & -\mathrm{Im}\,\mathbf{\Sigma} \\ \mathrm{Im}\,\mathbf{\Sigma} & \mathrm{Re}\,\mathbf{\Sigma} \end{bmatrix}.$$

Throughout the paper all vectors are column vectors. The symbols $'$ and * denote tranposition of a matrix and the hermitian operation, respectively. Given $\underline{X}_1, \ldots, \underline{X}_K$ independently identically complex normally distributed N-dimensional random vectors with expected value $\underline{0}$ and covariance matrix $\mathbf{\Sigma}$. Then, the $(N \times N)$ random matrix $\mathbf{W} = \sum_{k=1}^{K} \underline{X}_k \underline{X}_k^*$ is complex Wishart distributed with K degrees of freedom and parameter matrix $\mathbf{\Sigma}$: $\mathbf{W} \sim \mathcal{W}_N^{\mathbb{C}}(K, \mathbf{\Sigma})$. The probability density function (pdf) $f_{\mathbf{W}}$ of $\mathbf{W}$ and the characteristic function (cf) $\Phi_{\mathbf{W}}(\mathbf{\Theta})$ of the elements of $\mathbf{W}$ are known [4].

3. COMPLEX INVERSE WISHART DISTRIBUTION

Let the $(N \times N)$ matrix $\mathbf{W}$ be Wishart distributed: $\mathbf{W} \sim \mathcal{W}_N^{\mathbb{C}}(K, \mathbf{\Sigma})$. Then, the $(N \times N)$ matrix $\mathbf{W}^{-1}$ is complex inverse Wishart distributed with K degrees of freedom and parameter matrix $\mathbf{\Sigma}^{-1}$: $\mathbf{W}^{-1} \sim \mathcal{W}_N^{\mathbb{C}^{-1}}\left(K, \mathbf{\Sigma}^{-1}\right)$. In [1] the pdf of the real inverse Wishart distribution is calculated. In the following part a derivation of the distribution of the pdf of the complex inverse Wishart distribution is presented. The elements of $\mathbf{W}^{-1}$ are denoted by W^{ij} $(i, j = 1, \ldots, N)$. The transformation rule for pdfs is given by

$$f_{\mathbf{W}^{-1}}(\mathbf{W}^{-1}) = \left|\det\left[\frac{\partial \underline{v}}{\partial \underline{w}}\right]\right| f_{\mathbf{W}}(\mathbf{W}^{-1}),$$

where

$$\begin{aligned}\underline{v} \;=\; &\left(W^{11}, \ldots, W^{NN}, \mathrm{Re}\,W^{12}, \mathrm{Im}\,W^{12}, \ldots,\right.\\ &\left.\mathrm{Re}\,W^{N-1,N}, \mathrm{Im}\,W^{N-1,N}\right)'\end{aligned}$$

and

$$\underline{w} = (W_{11},\ldots,W_{NN},\operatorname{Re}W_{12},\operatorname{Im}W_{12},\ldots, \operatorname{Re}W_{N-1,N},\operatorname{Im}W_{N-1,N})'.$$

Using the formulae $\frac{\partial \mathbf{W}^{-1}}{\partial\theta} = -\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\theta}\mathbf{W}^{-1}$ and

$$\operatorname{vec}\left(\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\theta}\mathbf{W}^{-1}\right) = \left(\mathbf{W}^{-1'}\otimes\mathbf{W}^{-1}\right)\operatorname{vec}\left(\frac{\partial\mathbf{W}}{\partial\theta}\right),$$

leads to

$$\left|\det\left[\frac{\partial\underline{v}}{\partial\underline{w}}\right]\right| = 2^{-N(N-1)/2}\det\left(\mathbf{W}^{-1'}\otimes\mathbf{W}^{-1}\right)\cdot \left|\det\left[\operatorname{vec}\left(\frac{\partial\mathbf{W}}{\partial w_1}\right),\ldots,\operatorname{vec}\left(\frac{\partial\mathbf{W}}{\partial w_{N^2}}\right)\right]\right|.$$

We obtain for $\frac{\partial\mathbf{W}}{\partial\theta}$

$$\frac{\partial\mathbf{W}}{\partial\theta} = \begin{cases} \underline{e}_i\underline{e}_j' & : \text{ for } \theta = W_{ii} \\ \underline{e}_i\underline{e}_j' + \underline{e}_j\underline{e}_i' & : \text{ for } \theta = \operatorname{Re}W_{ij} \\ j(\underline{e}_i\underline{e}_j' - \underline{e}_j\underline{e}_i') & : \text{ for } \theta = \operatorname{Im}W_{ij} \end{cases}.$$

$\underline{e}_i$ is a real N dimensional unit vector. Therefore, it can be shown that

$$\left|\det\left[\operatorname{vec}\left(\frac{\partial\mathbf{W}}{\partial w_1}\right),\ldots,\operatorname{vec}\left(\frac{\partial\mathbf{W}}{\partial w_{N^2}}\right)\right]\right| = 2^{N(N-1)/2}.$$

Using $\det\left(\mathbf{W}^{-1'}\otimes\mathbf{W}^{-1}\right) = \det\left(\mathbf{W}^{-1'}\right)^N\det\left(\mathbf{W}^{-1}\right)^N$ and the pdf of the complex Wishart distribution gives for $\mathbf{W}^{-1}$ positiv definite

$$f_{\mathbf{W}^{-1}}(\mathbf{W}^{-1}) = \frac{\left(\det\mathbf{W}^{-1}\right)^{K+N}}{I(\mathbf{\Sigma})}\exp\left(-\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W}^{-1}\right)\right)$$

and $f_{\mathbf{W}^{-1}}(\mathbf{W}^{-1}) = 0$ for $\mathbf{W}^{-1}$ not positiv definite, with

$$I(\mathbf{\Sigma}) = \pi^{N(N-1)/2}\prod_{n=1}^{N}\Gamma(K-n+1)\left(\det\mathbf{\Sigma}\right)^K.$$

4. MOMENTS OF WISHART DISTRIBUTED MATRICES

For real random vectors it is well known that the cf can be k times continuously differentiated if the corresponding moments exist. Multiplying the kth derivative of the cf by j^{-k} and evaluating the resulting expression for argument zero gives the correponding kth moment. But in this paper, we are interested in moments of the complex elements of $\mathbf{S}$. The differentiation of $\Phi_{\mathbf{S}}(\mathbf{\Theta})$ with respect to θ_{ij} for $i \neq j$ is not allowed because $\mathbf{\Theta}$ is a hermitian matrix and the Cauchy-Riemannian differential equations are not satisfied if the element θ_{ji} is complex differentiated with respect to θ_{ij}. The following part shows that the differentiation of the cf $\Phi_{\mathbf{S}}(\mathbf{\Theta})$ with respect to elements θ_{ij} is possible if real and imaginary parts are treated separately.

We start by showing that the function $\operatorname{tr}(\mathbf{S\Theta})$ in

$$\Phi_{\mathbf{S}}(\mathbf{\Theta}) = \operatorname{E}\exp\left(j\operatorname{tr}(\mathbf{S\Theta})\right) \tag{1}$$

is real. Using $S_{ij} = S_{ij}^R + jS_{ij}^I$ and $\theta_{ij} = \theta_{ij}^R + j\theta_{ij}^I$ gives

$$\begin{aligned}\operatorname{tr}(\mathbf{S\Theta}) &= \sum_{j,k=1}^{N} S_{jk}\theta_{kj} \\ &= \sum_{j=1}^{N} S_{jj}^R\theta_{jj}^R + 2\sum_{\substack{j<k\\ j,k=1}}^{N}\left(S_{jk}^R\theta_{jk}^R + S_{jk}^I\theta_{jk}^I\right),\end{aligned} \tag{2}$$

where ^{R} and ^{I} denote the real and imaginary part, respectively. Equation (2) can be used to determine the derivative of $\Phi_{\mathbf{S}}(\mathbf{\Theta})$ with respect to the real and the imaginary part of θ_{ik}:

$$\left.\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ik}^R}\right|_{\mathbf{\Theta}=\mathbf{0}} = j\begin{cases}\operatorname{E}S_{ii}^R, & i=k \\ 2\operatorname{E}S_{ik}^R, & i\neq k\end{cases} \tag{3}$$

$$,\left.\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ik}^I}\right|_{\mathbf{\Theta}=\mathbf{0}} = j\begin{cases}0, & i=k \\ 2\operatorname{E}S_{ik}^I, & i\neq k\end{cases}. \tag{4}$$

This leads to

$$\left[\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ik}^R} + j\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ik}^I}\right]_{\mathbf{\Theta}=\mathbf{0}} = j\operatorname{E}S_{ik}\begin{cases}1, & i=k \\ 2, & i\neq k\end{cases}. \tag{5}$$

The cf (1) can be differentiated with respect to the real and the imaginary part of θ_{ik}. Let ν a real parameter of the matrix $\mathbf{A}$ then:

$$\frac{\partial|\mathbf{A}|}{\partial\nu} = |\mathbf{A}|\operatorname{tr}\left(\mathbf{A}^{-1}\frac{\partial\mathbf{A}}{\partial\nu}\right). \tag{6}$$

Using (6) we obtain

$$\begin{aligned}&\left[\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ik}^R} + j\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ik}^I}\right]_{\mathbf{\Theta}=\mathbf{0}} \\ &= j\Phi_{\mathbf{S}}(\mathbf{\Theta})\operatorname{tr}\left[\left(\mathbf{\Sigma}^{-1} - j\frac{1}{K}\mathbf{\Theta}\right)^{-1}\left(\frac{\mathbf{\Theta}}{\partial\theta_{ik}^R} + j\frac{\partial\mathbf{\Theta}}{\partial\theta_{ik}^I}\right)\right]\Bigg|_{\mathbf{\Theta}=\mathbf{0}} \\ &= j\Sigma_{ki}\begin{cases}1, & i=k \\ 2, & i\neq k\end{cases}.\end{aligned} \tag{7}$$

Combining (5) and (7) gives the first moment $\operatorname{E}S_{ik} = \Sigma_{ki}$. A similar approach can be used to obtain the second moment. The procedure can be significantly simplified by introduction of a formal differentiation:

$$\frac{\partial\mathbf{\Theta}}{\partial\theta_{ik}} = \underline{e}_i\underline{e}_k'. \tag{8}$$

When using (8) we do not mean complex differentiation, θ_{ik} and θ_{ki} are regarded as two different variables, see also [2]. We can simply write

$$\begin{aligned}\operatorname{E}S_{ik} &= \left.\frac{1}{j}\frac{\partial\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ki}}\right|_{\mathbf{\Theta}=\mathbf{0}} \\ &= \operatorname{E}\left(S_{ki}\exp\left(j\operatorname{tr}(\mathbf{S\Theta})\right)\right)\Big|_{\mathbf{\Theta}=\mathbf{0}} = \Sigma_{ik}.\end{aligned} \tag{9}$$

For higher order moments we obtain the following results.

Second Moment:

$$\operatorname{E}S_{ij}S_{kl} = -\left.\frac{\partial^2\Phi_{\mathbf{S}}(\mathbf{\Theta})}{\partial\theta_{ji}\partial\theta_{lk}}\right|_{\mathbf{\Theta}=\mathbf{0}} = \Sigma_{ij}\Sigma_{kl} + \frac{1}{K}\Sigma_{kj}\Sigma_{il}.$$

Third Moment:

$$\mathrm{E}\, S_{ij} S_{kl} S_{mn} = -\frac{1}{j} \frac{\partial^3 \Phi_{\mathbf{S}}(\Theta)}{\partial\theta_{ji}\partial\theta_{lk}\partial\theta_{nm}}\bigg|_{\Theta=0} =$$

$$\Sigma_{ij}\Sigma_{kl}\Sigma_{mn} + \frac{1}{K}\left(\Sigma_{mj}\Sigma_{in}\Sigma_{kl} + \Sigma_{ml}\Sigma_{kn}\Sigma_{ij} + \Sigma_{kj}\Sigma_{il}\Sigma_{mn}\right) + \frac{1}{K^2}\left(\Sigma_{kj}\Sigma_{in}\Sigma_{ml} + \Sigma_{kn}\Sigma_{mj}\Sigma_{il}\right).$$

Fourth Moment:

$$\mathrm{E}\, S_{ij} S_{kl} S_{mn} S_{op} = \frac{\partial^4 \Phi_{\mathbf{S}}(\Theta)}{\partial\Theta_{ji}\partial\Theta_{lk}\partial\Theta_{nm}\partial\Theta_{po}}\bigg|_{\Theta=0} =$$

$$\Sigma_{ij}\Sigma_{kl}\Sigma_{nm}\Sigma_{op} + \frac{1}{K}\left(\Sigma_{oj}\Sigma_{ip}\Sigma_{kl}\Sigma_{mn} + \Sigma_{ij}\Sigma_{ol}\Sigma_{kp}\Sigma_{mn} + \Sigma_{ij}\Sigma_{kl}\Sigma_{on}\Sigma_{mp} + \Sigma_{op}\Sigma_{mj}\Sigma_{in}\Sigma_{kl} + \Sigma_{op}\Sigma_{ml}\Sigma_{kn}\Sigma_{ij} + \Sigma_{op}\Sigma_{mn}\Sigma_{il}\Sigma_{kj}\right) + O\left(\frac{1}{K^2}\right)$$

Using these relations the moments of complex Wishart distributed matrices up to order four can be easily derived. The following formulae are examples:

$$\mathrm{E}\,\mathbf{S} = \mathbf{\Sigma} \tag{10}$$

$$\mathrm{E}\ \mathrm{tr}(\mathbf{ASBS}) = \mathrm{tr}(\mathbf{A\Sigma B\Sigma}) + \frac{1}{K}\mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma}) \tag{11}$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{AS})\,\mathrm{tr}(\mathbf{BS})\right) = \mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma}) + \frac{1}{K}\mathrm{tr}(\mathbf{A\Sigma B\Sigma}) \tag{12}$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{ASBS})\,\mathrm{tr}(\mathbf{CS})\right) = \mathrm{tr}(\mathbf{A\Sigma B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma}) + \frac{1}{K}\Big(\mathrm{tr}(\mathbf{A\Sigma B\Sigma C\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma C\Sigma B\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma})\Big) + O\left(\frac{1}{K^2}\right) \tag{13}$$

$$\mathrm{E}\ \mathrm{tr}(\mathbf{ASBSCSDS}) = \mathrm{tr}(\mathbf{A\Sigma B\Sigma C\Sigma D\Sigma}) + \frac{1}{K}\Big(\mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma C\Sigma D\Sigma}) + \mathrm{tr}(\mathbf{B\Sigma})\,\mathrm{tr}(\mathbf{A\Sigma C\Sigma D\Sigma}) + \mathrm{tr}(\mathbf{C\Sigma})\,\mathrm{tr}(\mathbf{A\Sigma B\Sigma D\Sigma}) + \mathrm{tr}(\mathbf{D\Sigma})\,\mathrm{tr}(\mathbf{A\Sigma B\Sigma C\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma D\Sigma}) + \mathrm{tr}(\mathbf{B\Sigma C\Sigma})\,\mathrm{tr}(\mathbf{D\Sigma A\Sigma})\Big) + O\left(\frac{1}{K^2}\right) \tag{14}$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{ASBS})\,\mathrm{tr}(\mathbf{CSDS})\right) = \mathrm{tr}(\mathbf{A\Sigma B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma D\Sigma}) + \frac{1}{K}\Big(\mathrm{tr}(\mathbf{A\Sigma B\Sigma C\Sigma D\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma C\Sigma D\Sigma B\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma B\Sigma D\Sigma C\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma D\Sigma C\Sigma B\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma})\,\mathrm{tr}(\mathbf{D\Sigma}) + \mathrm{tr}(\mathbf{C\Sigma D\Sigma})\,\mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma})\Big) + O\left(\frac{1}{K^2}\right) \tag{15}$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{AS})\,\mathrm{tr}(\mathbf{BS})\,\mathrm{tr}(\mathbf{CSDS})\right) = \mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma D\Sigma}) + \frac{1}{K}\Big(\mathrm{tr}(\mathbf{A\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma})\,\mathrm{tr}(\mathbf{C\Sigma})\,\mathrm{tr}(\mathbf{D\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma C\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma D\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma C\Sigma D\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma}) + \mathrm{tr}(\mathbf{A\Sigma D\Sigma C\Sigma})\,\mathrm{tr}(\mathbf{B\Sigma}) + \mathrm{tr}(\mathbf{B\Sigma C\Sigma D\Sigma})\,\mathrm{tr}(\mathbf{A\Sigma}) + \mathrm{tr}(\mathbf{B\Sigma D\Sigma C\Sigma})\,\mathrm{tr}(\mathbf{A\Sigma})\Big) + O\left(\frac{1}{K^2}\right) \tag{16}$$

5. MOMENTS OF COMPLEX INVERSE WISHART DISTRIBUTED MATRICES

The moments of inverse complex Wishart distributed random matrices can be calculated without knowledge of the pdf or the cf. In [5] and [6] an identity is given by which moments of real Wishart and inverse real Wishart distributed random matrices can be calculated. In this paper we use this identity. We derive a simplified form of the identity for the calculation of the moments of interest. We define the real matrices

$$\mathbf{W}_R = \begin{bmatrix} \mathrm{Re}\,\mathbf{W} & -\mathrm{Im}\,\mathbf{W} \\ \mathrm{Im}\,\mathbf{W} & \mathrm{Re}\,\mathbf{W} \end{bmatrix},$$

$$\mathbf{W}_R^{-1} = \begin{bmatrix} \mathrm{Re}\,\mathbf{W}^{-1} & -\mathrm{Im}\,\mathbf{W}^{-1} \\ \mathrm{Im}\,\mathbf{W}^{-1} & \mathrm{Re}\,\mathbf{W}^{-1} \end{bmatrix},$$

and denote the elements of $\mathbf{W}_R$ and $\mathbf{W}_R^{-1}$ by $W_{R,ij}$ and W_R^{ij} $(i,j = 1,\ldots,2N)$, respectively. Then, the following corollary can be proven [7].

Corollary: Let all moments of the complex random variables $W^{a_1 b_1}$ $(a_1, b_1 = 1,\ldots,N)$ exist up to order $J+1$. The function $g : C^{N(N+1)/2} \to C$ is defined by $g = \prod_{i=1}^{I} W^{a_i b_i}$ with $I \leq J$. If real and imaginary part of g fulfill the conditions of Stokes theorem as described by theorem 2.1 in [6] then:

$$\mathrm{E}\left(g\Sigma^{ba}\right) = -\sum_{j=1}^{I}\prod_{\substack{i=1\\i\neq j}}^{I} W_R^{a_i b_i} W_R^{a_j a} W_R^{b b_j} + (K-N)\,\mathrm{E}\left(g W^{ba}\right). \tag{17}$$

Using relation (17) we can determine the moments of complex inverse Wishart distributed matrices.

First Moment: Use equation (17) with $g = 1$, $a = i$, $b = j$:

$$\mathrm{E}\, W^{ji} = \frac{1}{K-N}\Sigma^{ji}. \tag{18}$$

Second Moment: Using $g = W^{ij}$ and $a = k$, $b = l$ leads to

$$\frac{1}{K-N}\Sigma^{ij}\Sigma^{lk} = -\,\mathrm{E}\, W^{ik}W^{lj} + (K-N)\,\mathrm{E}\, W^{ij}W^{lk}. \tag{19}$$

For $g = W^{lj}$ and $a = k$, $b = i$ one obtains

$$\frac{1}{K-N}\Sigma^{lj}\Sigma^{ik} = -\,\mathrm{E}\, W^{lk}W^{ij} + (K-N)\,\mathrm{E}\, W^{lj}W^{ik}. \tag{20}$$

Relations (19) and (20) constitute a linear system of equations with the two unknowns $\mathrm{E}\, W^{lk}W^{ij}$ and $\mathrm{E}\, W^{lj}W^{ik}$. The solution for $EW^{ij}W^{lk}$ is given by

$$\mathrm{E}\, W^{ij}W^{lk} = \frac{\Sigma^{ij}\Sigma^{lk} + \frac{1}{K-N}\Sigma^{lj}\Sigma^{ik}}{(K-N)^2 - 1}.$$

For an integer x and $(K-N) >> y$ we can use

$$\frac{1}{(K-N)^x\Big((K-N)^2-y\Big)} = \frac{1}{(K-N)^{x+2}} + O\left(\frac{1}{(K-N)^{x+4}}\right) \quad (21)$$

and write

$$\mathrm{E}\, W^{ij}W^{lk} = \frac{1}{(K-N)^2}\Sigma^{ij}\Sigma^{lk} + \frac{1}{(K-N)^3}\Sigma^{lj}\Sigma^{ik} + O\left(\frac{1}{(K-N)^4}\right). \quad (22)$$

Third and Fourth Moment: The procedure used for derivation of the second moment can be applied for the calculation of the third and fourth moment by appropriate choice of g and a, b. Then a system of 4 and 12 equations with 5 and 12 unknowns is obtained, respectively [8]. The solution of these linear equation systems gives:

$$\mathrm{E}\, W^{km}W^{nj}W^{il} = \frac{1}{(K-N)^3}\Sigma^{km}\Sigma^{nj}\Sigma^{il} + \frac{1}{(K-N)^4}\left(\Sigma^{kj}\Sigma^{nm}\Sigma^{il} + \Sigma^{nj}\Sigma^{kl}\Sigma^{im} + \Sigma^{ij}\Sigma^{km}\Sigma^{nl}\right) + O\left(\frac{1}{(K-N)^5}\right). \quad (23)$$

$$\mathrm{E}\, W^{ip}W^{oj}W^{kn}W^{ml} = \frac{1}{(K-N)^4}\Sigma^{ip}\Sigma^{oj}\Sigma^{kn}\Sigma^{mp} + \frac{1}{(K-N)^5}\Big(\Sigma^{ip}\Sigma^{on}\Sigma^{kj}\Sigma^{ml} + \Sigma^{ip}\Sigma^{kn}\Sigma^{ol}\Sigma^{mj} + \Sigma^{ip}\Sigma^{mn}\Sigma^{oj}\Sigma^{kl} + \Sigma^{in}\Sigma^{kp}\Sigma^{oj}\Sigma^{ml} + \Sigma^{il}\Sigma^{oj}\Sigma^{kn}\Sigma^{mp} + \Sigma^{op}\Sigma^{kn}\Sigma^{ij}\Sigma^{ml}\Big) + O\left(\frac{1}{(K-N)^6}\right) \quad (24)$$

Using the first moment (18) and the approximations of the second to fourth moment (22) to (24) the following identities can be derived [8].

$$\mathrm{E}\,\mathbf{S}^{-1} = \frac{K}{K-N}\mathbf{\Sigma}^{-1} \quad (25)$$

$$\mathrm{E}\ \mathrm{tr}(\mathbf{AS}^{-1}\mathbf{BS}^{-1}) = \frac{K^2}{(K-N)^2}\mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1}) + \frac{K^2}{(K-N)^3}\mathrm{tr}(\mathbf{A\Sigma}^{-1})\,\mathrm{tr}(\mathbf{B\Sigma}^{-1}) + O\left(\frac{K^2}{(K-N)^4}\right) \quad (26)$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{AS}^{-1})\,\mathrm{tr}(\mathbf{BS}^{-1})\right) = \frac{K^2}{(K-N)^2}\mathrm{tr}(\mathbf{A\Sigma}^{-1})\,\mathrm{tr}(\mathbf{B\Sigma}^{-1}) + \frac{K^2}{(K-N)^3}\mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1}) + O\left(\frac{K^2}{(K-N)^4}\right) \quad (27)$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{AS}^{-1}\mathbf{BS}^{-1})\,\mathrm{tr}(\mathbf{CS}^{-1})\right) = \frac{K^3}{(K-N)^3}\mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1})\,\mathrm{tr}(\mathbf{C\Sigma}^{-1}) + \frac{K^3}{(K-N)^4}\Big(\mathrm{tr}(\mathbf{A\Sigma}^{-1})\,\mathrm{tr}(\mathbf{B\Sigma}^{-1})\,\mathrm{tr}(\mathbf{C\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{C\Sigma}^{-1}\mathbf{B\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1}\mathbf{C\Sigma}^{-1})\Big) + O\left(\frac{K^3}{(K-N)^5}\right) \quad (28)$$

$$\mathrm{E}\left(\mathrm{tr}(\mathbf{AS}^{-1}\mathbf{BS}^{-1})\,\mathrm{tr}(\mathbf{CS}^{-1}\mathbf{DS}^{-1})\right) = \frac{K^4}{(K-N)^4}\mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1})\,\mathrm{tr}(\mathbf{C\Sigma}^{-1}\mathbf{D\Sigma}^{-1}) + \frac{K^4}{(K-N)^5}\Big(\mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1}\mathbf{C\Sigma}^{-1}\mathbf{D\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1}\mathbf{D\Sigma}^{-1}\mathbf{C\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{C\Sigma}^{-1}\mathbf{D\Sigma}^{-1}\mathbf{B\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{D\Sigma}^{-1}\mathbf{C\Sigma}^{-1}\mathbf{B\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1}\mathbf{B\Sigma}^{-1})\,\mathrm{tr}(\mathbf{C\Sigma}^{-1})\,\mathrm{tr}(\mathbf{D\Sigma}^{-1}) + \mathrm{tr}(\mathbf{A\Sigma}^{-1})\,\mathrm{tr}(\mathbf{B\Sigma}^{-1})\,\mathrm{tr}(\mathbf{C\Sigma}^{-1}\mathbf{D\Sigma}^{-1})\Big) + O\left(\frac{K^4}{(K-N)^6}\right) \quad (29)$$

REFERENCES

[1] T. W. Anderson. *An Introduction to Multivariate Statistical Analysis.* John Wiley & Sons, New York, 1984.

[2] D. H. Brandwood. A complex gradient operator and its application in adaptive array theory. *Proc. IEE*, 130, Pts. F and H(1):11–16, February 1983.

[3] P. Forster and G. Vezzosi. Approximate ML after spheroidal beamforming in the narrow band and wide band cases. In *Proc. 5th European Signal Processing Conf., Signal Processing V: Theories and Applications*, pages 307–310, Grenoble, 1988.

[4] N. R. Goodman. Statistical analysis based on a certain multivariate complex gaussian distribution (an intoduction). *Ann. Math. Statist.*, 34:152–176, 1963.

[5] L. R. Haff. Minimax estimators for a multinormal precision matrix. *J. Multivariate Anal.*, 7:374–385, 1977.

[6] L. R. Haff. An identity for the wishart distribution with application. *J. Multivariate Anal.*, 9:531 – 544, 1979.

[7] D. Kraus. Approximative Maximum–Likelihood–Schätzung und verwandte Verfahren zur Ortung und Signalschätzung mit Sensorgruppen (in German). Doctoral thesis, Ruhr-Universität Bochum, 1993.

[8] D. Maiwald. Untersuchung eines Kleinste-Quadrate- bzw. eines approximativen Maximum-Likelihood-Zweistufenschätzers zur Analyse stationärer Wellenfelder (in German). Master's thesis, Ruhr-Universität Bochum, 1989.