

PARAMETER ESTIMATION OF TWO-DIMENSIONAL MOVING AVERAGE RANDOM FIELDS: ALGORITHMS AND BOUNDS *

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ABSTRACT

This paper considers the problem of estimating the parameters of two-dimensional moving average random fields. We first address the problem of expressing the covariance matrix of a moving average random field, in terms of the model parameters. Assuming the random field is Gaussian, we derive a closed form expression for the Cramer-Rao lower bound on the error variance in jointly estimating the model parameters. A computationally efficient algorithm for estimating the parameters of the moving average model is developed. The algorithm initially fits a two-dimensional autoregressive model to the observed field, then uses the estimated parameters to compute the moving average model.

1. INTRODUCTION

The problem of estimating the parameters of a two-dimensional (2-D) real valued discrete and homogeneous moving average random field, from a single observed realization of the field is of great theoretical and practical importance. For example, it arises quite naturally in terms of estimating the parameters of the purely indeterministic component of natural textures in images [6], as well as in image segmentation and restoration problems.

More specifically, in [6] we have presented a texture model which is based on the 2-D Wold-type decomposition of homogeneous random fields, [2]. In this framework, the texture field is assumed to be a realization of a regular homogeneous random field, which can have a mixed spectral distribution. The texture is represented as a sum of purely indeterministic, harmonic, and a countable number of evanescent fields. The harmonic and evanescent components of the field result in the structural attributes of the observed realization, while the purely indeterministic component is the structureless, "random looking" component of the texture field.

The general problem of estimating the parameters of random fields has received considerable attention. Most approaches for estimating the parameters of purely indeterministic random fields, concentrate on fitting 2-D autoregressive (AR) models to the observed field. Least squares solution of the set of 2-D normal equations is a method widely used in various image processing applications like image restoration and segmentation. A Levinson-type algorithm for solving the set of 2-D normal equations of a continuous support non-symmetrical half plane (NSHP) AR model is derived in [5]. The asymptotic Cramer-Rao bound

for the parameters of a Gaussian purely indeterministic field was derived by Whittle [4].

In this paper we propose a computationally efficient algorithm for estimating the parameters of MA random fields using a *finite dimension, single observed realization* of this field. The algorithm is an extension to two-dimensions of Durbin's "MA by AR" method, [1], for estimating the parameters of scalar moving average processes. The algorithm has two stages. In the first stage, a two-dimensional NSHP AR model is fit to the observed field, using a least squares solution of the 2-D normal equations, or alternatively by using a finite support version of Marzetta's [5] Levinson type algorithm. In the second stage, the estimated parameters of the AR model are used to compute the parameters of the moving average model, through a least squares solution of a system of linear equations. The overall algorithm is computationally efficient.

We also address here, the problem of expressing the covariance matrix of the observed field in terms of the MA model parameters. Then, assuming the MA field is Gaussian, we employ this result to establish bounds on the achievable accuracy in jointly estimating the parameters of the MA modeled purely indeterministic random field. We derive closed form exact expression for the Cramer Rao lower bound on the achievable estimation accuracy. Using the expressions of the covariance matrix in terms of the MA model parameters, we then derive a maximum likelihood algorithm for these. The previously derived "MA by AR" algorithm is used for initialization of the multi-dimensional search involved in the maximum likelihood estimation algorithm.

The paper is organized as follows. In section 2 we consider the problem of representing the covariance matrix of the observed MA field in terms of the MA model parameters. In section 3 a closed form expression for the CRB on the error variance in jointly estimating the MA model parameters, is derived. In section 4 we develop the computationally efficient "MA by AR" estimation algorithm. In section 5 we present some numerical examples.

2. THE PARAMETRIC REPRESENTATION OF THE MA FIELD AND ITS COVARIANCE MATRIX

Let $\{y(n, m), (n, m) \in \mathcal{Z}^2\}$, be a real valued, purely-indeterministic, homogeneous random field. Then, [3], $y(n, m)$ can be uniquely represented by

$$y(n, m) = \sum_{(0,0) \preceq (k,\ell)} b(k, \ell) u(n - k, m - \ell) \quad (1)$$

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where $\sum_{(0,0) \leq (k,\ell)} b^2(k,\ell) < \infty$; $b(0,0) = 1$, and $\{u(n,m)\}$ is the innovations field of $\{y(n,m)\}$ with respect to the defined total-order definition. $\{u(n,m)\}$ is a white noise field. We therefore conclude that the most general model of any purely-indeterministic random field is the innovations driven, NSHP support MA model (1).

In practice, the observed random field is of finite dimensions. Hence, let $\{y(n,m), (n,m) \in D\}$ where $D = \{(i,j) | 0 \leq i \leq S-1, 0 \leq j \leq T-1\}$ be the observed random field. The MA model (1) is, in general, of infinite dimensions. In this paper we restrict our attention to MA models of finite dimensional NSHP support. Next, we elaborate on expressing the covariance matrix of the observed 2-D MA random field in terms of the model parameters.

Assumption 1: The purely-indeterministic field is a real valued MA field, whose model is given by (1) with $(k,\ell) \in S_{N,M}$, where $S_{N,M} = \{(i,j) | i = 0, 0 \leq j \leq M\} \cup \{(i,j) | 1 \leq i \leq N, -M \leq j \leq M\}$, and N, M are *a-priori* known. The driving noise of the MA model is a zero mean, real valued white noise field with variance σ^2 . Thus, (1) is replaced by

$$y(n,m) = \sum_{(k,\ell) \in S_{N,M}} b(k,\ell) u(n-k, m-\ell). \quad (2)$$

The parameter vector of the observed field $\{y(n,m)\}$ is given by

$$\theta = \left[\sigma^2, b(0,1), \dots, b(0,M), b(1,-M), \dots, b(1,M), \dots, b(N,-M), \dots, b(N,M) \right]^T. \quad (3)$$

Let $\mathbf{0}_k$ denote a k -dimensional row vector of zeros. Let also,

$$\begin{aligned} \mathbf{b}_0 &= [\mathbf{0}_M, 1, b(0,1), \dots, b(0,M), \mathbf{0}_{T-1}], \\ \mathbf{b}_1 &= [b(1,-M), \dots, b(1,0), \dots, b(1,M), \mathbf{0}_{T-1}], \\ &\vdots \\ \mathbf{b}_N &= [b(N,-M), \dots, b(N,0), \dots, b(N,M)], \end{aligned} \quad (4)$$

and

$$\mathbf{b} = [\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_N]. \quad (5)$$

Note that \mathbf{b} is a $(T+2M) \cdot (N+1) - (T-1)$ dimensional row vector.

Define the following $T \times (T+2M) \cdot (N+1)$ banded Toeplitz matrix

$$[\tilde{\mathbf{B}}]_{i,j} = \begin{cases} \mathbf{b}(j-i+1), & j \geq i \\ 0, & i < j \end{cases}; \quad (6)$$

where $\mathbf{b}(i) = 0$ for $i < 0$ and $i > (T+2M) \cdot (N+1) - (T-1)$. Finally, let \mathbf{B} be the following $ST \times (T+2M)(S+N)$ block matrix

$$\begin{bmatrix} \tilde{\mathbf{B}} & \mathbf{0}_{T \times (T+2M)} & \dots & \mathbf{0}_{T \times (T+2M)} \\ \mathbf{0}_{T \times (T+2M)} & \tilde{\mathbf{B}} & \dots & \mathbf{0}_{T \times (T+2M)} \\ & \ddots & \ddots & \ddots \\ \mathbf{0}_{T \times (T+2M)} & \dots & \tilde{\mathbf{B}} & \mathbf{0}_{T \times (T+2M)} \\ \mathbf{0}_{T \times (T+2M)} & \dots & \mathbf{0}_{T \times (T+2M)} & \tilde{\mathbf{B}} \end{bmatrix} \quad (7)$$

Hence the covariance matrix of the observed field is given in terms of the MA model parameters by

$$\Gamma = \sigma^2 \mathbf{B} \mathbf{B}^T. \quad (8)$$

Note that (7)-(8) can be made valid for any type of support of the MA model, simply by redefining \mathbf{b} and \mathbf{B} .

3. THE CRB ON THE MA MODEL PARAMETERS

Assume that the driving noise of the NSHP MA model is a zero mean, real valued Gaussian white noise field with variance σ^2 . Hence the observed field $\{y(n,m)\}$ is also Gaussian. The general expression for the Fisher Information Matrix (FIM) of a real, zero mean, Gaussian process is given by

$$[\mathbf{J}(\theta)]_{i,j} = \frac{1}{2} \text{tr} \left\{ \Gamma^{-1} \frac{\partial \Gamma}{\partial \theta_i} \Gamma^{-1} \frac{\partial \Gamma}{\partial \theta_j} \right\}, \quad (9)$$

where Γ is the observation vector covariance matrix, and $[\mathbf{J}(\theta)]_{i,j}$ denotes the (i,j) entry of the matrix \mathbf{J} .

Note that

$$\frac{\partial \tilde{\mathbf{B}}}{\partial b(k,\ell)} = \tilde{\mathbf{U}}_{(k,\ell)} \quad (10)$$

where $\tilde{\mathbf{U}}_{(k,\ell)}$ is the up shift matrix

$$[\tilde{\mathbf{U}}_{(k,\ell)}]_{i,j} = \begin{cases} 1, & j - i = k(T+2M) + M + \ell \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Taking the partial derivatives of Γ with respect to the MA model parameters we have for all $(k,\ell) \in S_{N,M} \setminus \{(0,0)\}$

$$\frac{\partial \Gamma}{\partial b(k,\ell)} = \sigma^2 \left[\tilde{\mathbf{U}}_{(k,\ell)} \mathbf{B}^T + \mathbf{B} \tilde{\mathbf{U}}_{(k,\ell)}^T \right], \quad (12)$$

where $\tilde{\mathbf{U}}_{(k,\ell)}$ is the following $ST \times (T+2M)(S+N)$ block matrix

$$\begin{bmatrix} \tilde{\mathbf{U}}_{(k,\ell)} & \mathbf{0}_{T \times (T+2M)} & \dots & \mathbf{0}_{T \times (T+2M)} \\ \mathbf{0}_{T \times (T+2M)} & \tilde{\mathbf{U}}_{(k,\ell)} & \dots & \mathbf{0}_{T \times (T+2M)} \\ & \ddots & \ddots & \ddots \\ \mathbf{0}_{T \times (T+2M)} & \dots & \tilde{\mathbf{U}}_{(k,\ell)} & \mathbf{0}_{T \times (T+2M)} \\ \mathbf{0}_{T \times (T+2M)} & \dots & \mathbf{0}_{T \times (T+2M)} & \tilde{\mathbf{U}}_{(k,\ell)} \end{bmatrix} \quad (13)$$

Also,

$$\frac{\partial \Gamma}{\partial \sigma^2} = \frac{1}{\sigma^2} \Gamma. \quad (14)$$

Substituting (8), (12), and (14) into (9) we obtain a closed form expression for the FIM of 2-D Gaussian MA random fields.

4. 2-D MOVING AVERAGE PARAMETER ESTIMATION

The parameter estimation algorithm which we present in this section is an extension to two-dimensions of the algorithm proposed by Durbin, [1], for estimating the parameters of scalar MA processes. The idea is to fit a NSHP AR model to the observed field, and then using the estimated AR parameters to estimate the MA model parameters.

It was shown by Whittle, [4], that any purely-indeterministic random field whose spectral density is analytic in some neighborhood of the unit bicircle and strictly positive on the unit bicircle, can be represented by a NSHP AR model, of generally infinite dimensions. This result was later extended and was shown to hold even under milder conditions, [3]. Hence, any 2-D purely-indeterministic moving average random field that satisfies the foregoing conditions can be fit with a NSHP AR model. Since parameter estimation algorithms of 2-D AR random fields are available (e.g., [5]), we employ such an algorithm as the first step of the proposed procedure for estimating the parameters of the MA field.

Let $S_{P,Q}$ be defined similarly to $S_{N,M}$, and let $S_{P,Q} \setminus \{(0,0)\}$ be the NSHP support of the MA field AR model. In general, $S_{P,Q}$ is of infinite dimensions. In practice, we must choose finite values for P and Q , and hence an approximation error is introduced. It is obvious that such a method is necessarily inconsistent, even if the covariance function of the observed field is *a-priori* known, since no MA field can be exactly modeled by a finite support AR model. However the bias of the estimates can be made arbitrarily small by sufficiently increasing the support of the AR model, $S_{P,Q}$. Therefore, we choose P and Q such that $P \gg N$ and $Q \gg M$, i.e., the finite support of the AR model is chosen to be much larger than that of the MA model. More specifically, let the 2-D, finite support, MA model of the data be given by (2), and let the approximated, finite support, NSHP AR model of the same field be given by

$$y(n, m) = - \sum_{(k, \ell) \in S_{P,Q} \setminus \{(0,0)\}} a(k, \ell) y(n-k, m-\ell) + u(n, m). \quad (15)$$

Define $B(z_1, z_2) = \sum_{(k, \ell) \in S_{N,M}} b(k, \ell) z_1^{-k} z_2^{-\ell}$ and $A(z_1, z_2) = \sum_{(k, \ell) \in S_{P,Q}} a(k, \ell) z_1^{-k} z_2^{-\ell}$, where $a(0, 0) = 1$. We therefore have the approximate relation

$$A(z_1, z_2) B(z_1, z_2) \approx 1. \quad (16)$$

Let

$$\tilde{\mathbf{b}} = \begin{bmatrix} b(0, 1), \dots, b(0, M), b(1, -M), \dots, b(1, M), \dots, \\ b(N, -M), \dots, b(N, M) \end{bmatrix}^T. \quad (17)$$

Similarly, let

$$\mathbf{a}_0 = [a(0, 1), \dots, a(0, Q), 0_M]^T, \quad (18)$$

$$\mathbf{a}_1 = [a(1, -(Q-1)), \dots, a(1, 0), \dots, a(1, Q), 0_M]^T, \quad (19)$$

$$\mathbf{a}_k = [0_M, a(k, -Q), \dots, a(k, 0), \dots, a(k, Q), 0_M]^T \quad (20)$$

for $2 \leq k \leq P$, and let

$$\mathbf{a} = [\mathbf{a}_0^T, \mathbf{a}_1^T, \dots, \mathbf{a}_P^T, \mathbf{0}_{[2(Q+M)+1]N}^T]^T. \quad (21)$$

We can now set the following linear system of equations by equating the coefficients of identical powers of $z_1^{-k} z_2^{-\ell}$

$$\mathbf{A} \tilde{\mathbf{b}} + \mathbf{a} = \mathbf{e}, \quad (22)$$

where \mathbf{e} is the approximation error vector, and \mathbf{A} is the

following block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0^0 & \mathbf{W} & \mathbf{W} & \dots & \mathbf{W} \\ \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_0 & \mathbf{Z} & \dots & \mathbf{Z} \\ \mathbf{A}_2^0 & \mathbf{A}_1 & \mathbf{A}_0 & \dots & \mathbf{V} \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & \mathbf{V} \\ \mathbf{A}_N^0 & \mathbf{A}_{N-1} & \dots & \dots & \mathbf{A}_0 \\ \vdots & \vdots & & & \vdots \\ \mathbf{A}_P^0 & \mathbf{A}_{P-1} & \dots & \dots & \mathbf{A}_{P-N} \\ \mathbf{0}_{[2(Q+M)+1] \times M} & \mathbf{A}_P & \dots & \dots & \mathbf{A}_{P-N-1} \\ \mathbf{0}_{[2(Q+M)+1] \times M} & \mathbf{V} & \mathbf{A}_P & \dots & \mathbf{A}_{P-N-2} \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{0}_{[2(Q+M)+1] \times M} & \mathbf{V} & \dots & & \mathbf{A}_P \end{bmatrix}, \quad (23)$$

where we define $\mathbf{W} = \mathbf{0}_{(Q+M) \times (2M+1)}$, $\mathbf{Z} = \mathbf{0}_{(2Q+M) \times (2M+1)}$ and $\mathbf{V} = \mathbf{0}_{[2(Q+M)+1] \times (2M+1)}$. Each of the blocks of \mathbf{A} is a Toeplitz matrix. The structure of the blocks is given below.

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{0}_{Q \times (2M+1)} & & & \\ 1 & 0 & \dots & 0 \\ a(0, 1) & 1 & \dots & 0 \\ \vdots & & \ddots & 0 \\ \vdots & \vdots & & 1 \\ \vdots & & & \vdots \\ a(0, Q) & \dots & a(0, Q-2M) & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & a(0, Q) \end{bmatrix}, \quad (24)$$

and for $1 \leq k \leq P$,

$$\mathbf{A}_k = \begin{bmatrix} a(k, -Q) & 0 & \dots & 0 \\ a(k, -(Q-1)) & a(k, -Q) & \dots & 0 \\ \vdots & & \ddots & 0 \\ \vdots & \vdots & & a(k, -Q) \\ \vdots & & & \vdots \\ a(k, Q) & \dots & a(k, Q-2M) & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & a(k, Q) \end{bmatrix}. \quad (25)$$

Note that the matrices \mathbf{A}_k , $0 \leq k \leq P$, are all $[2(Q+M)+1] \times (2M+1)$ dimensional matrices. In addition, let $\tilde{\mathbf{A}}_0$ be the $(2Q+M) \times (2M+1)$ sub block of \mathbf{A}_0 consisting of its $2Q+M$ lower rows, and let \mathbf{A}_0^0 be the $(Q+M) \times M$ lower right sub block of \mathbf{A}_0 . Similarly, let $\tilde{\mathbf{A}}_k$, $1 \leq k \leq P$, be the $(2Q+M) \times M$ lower right sub block of \mathbf{A}_k , and

$$\mathbf{A}_k^0 = \begin{bmatrix} \mathbf{0}_{(M+1) \times M} \\ \tilde{\mathbf{A}}_k \end{bmatrix}. \quad (26)$$

The MA model parameters can now be found by minimizing the sum of the squared approximation error. The solution to this linear least squares problem is

$$\tilde{\mathbf{b}} = -(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{a}. \quad (27)$$

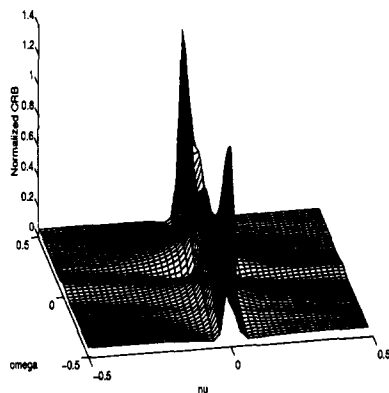


Figure 1. The ratio of the squared root of the CRLB to the spectral density function for the field whose parameters are listed in Table 1.

In the actual solution for the MA model parameters, the parameters of the AR model, $\{a(k, \ell)\}$, are replaced by their estimated values, obtained by solving the corresponding set of 2-D normal equations, using the *estimated* covariance function.

Finally, note that the proposed algorithm is derived using no *a-priori* assumptions regarding the probability density function of the observed field. It is therefore applicable to Gaussian moving average fields, as well as to non-Gaussian ones.

The main advantage of the "MA by AR" algorithm of the previous section is that it requires only the solution of two sets of linear systems of equations (one to estimate the AR parameters by solving the set of 2-D normal equations, and the second is the solution (27) to (22)). In particular there is no need for an iterative solution. However, the estimates are biased and inconsistent. Hence, improved estimation algorithms are required in cases where the performance of the "MA by AR" algorithm is not acceptable. The "MA by AR" algorithm can then serve to initialize a more sophisticated algorithm. One such estimator is the maximum likelihood estimator (MLE) for Gaussian moving average fields, [7].

5. NUMERICAL EXAMPLES

To gain more insight into the performance of the proposed algorithms we resort to numerical evaluation and Monte-Carlo simulations of some specific examples.

It is shown in [7] that the shape of the CR bound on the error variance in estimating the spectral density of the field as a function of frequency, matches the shape of the MA field spectral density. In order to further investigate the dependence of the CRB on the shape of the spectral density, we depict in Fig. 1 the normalized CRB, i.e., the ratio of the squared root of the CRB to the spectral density function of the MA field. We note that the lower bound on the error variance in estimating the MA field spectral density function is relatively higher in those frequency regions where the MA model transfer function is close zero. In other words, the estimation of the MA field spectral density function is less accurate in frequency regions where the spectral density function is close to zero than in regions of higher spectral density.

Table 1. Estimation results of the MA model parameters for different data sizes, using the "MA by AR" algorithm. The approximating NSHP AR model support is $S_{10,10}$.

| <i>Parameters</i> | | <i>S</i> = 30, <i>T</i> = 30 | | <i>S</i> = 100, <i>T</i> = 100 | |
|-------------------|------|------------------------------|---------|--------------------------------|---------|
| | | bias | std | bias | std |
| σ^2 | 1 | 0.00847 | 0.04759 | 0.05509 | 0.01644 |
| $b(0,1)$ | -0.9 | 0.16585 | 0.02303 | 0.08664 | 0.00523 |
| $b(1,-1)$ | 0.1 | -0.02929 | 0.03030 | -0.01317 | 0.00903 |
| $b(1,0)$ | -0.5 | 0.04523 | 0.03703 | 0.01500 | 0.01123 |
| $b(1,1)$ | 0.4 | -0.09762 | 0.03268 | -0.04679 | 0.00842 |

Consider the field whose parameters are listed in Table 1. Using the numerical results, we conclude that for the "MA by AR" algorithm, increasing the dimensions of the approximating AR model support reduces the bias of the estimated MA model parameters, and increases the standard deviation of the estimation error. It is shown in [7] that for small data sizes the ML algorithm is slightly biased. However, its bias is considerably lower than that of the "MA by AR" algorithm which is used for its initialization. By increasing the dimensions of the observed field, the bias of the estimates drops sharply, since the estimates of the covariance function are more accurate.

We therefore conclude that the bias of the "MA by AR" algorithm decreases when increasing the dimensions of the observed field, or the dimensions of the approximating NSHP AR model support. It is also demonstrated that the "MA by AR" estimator is a good choice for implementing the initialization phase of the maximum likelihood algorithm. Furthermore, as the data size increases, the maximum likelihood method becomes computationally prohibitive due to its heavy computational and storage requirements, while the computationally efficient "MA by AR" algorithm becomes less biased, and therefore offers an increasingly attractive alternative to ML estimation.

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