ESTIMATION OF TRANSFER FUNCTION PARAMETERS WITH OUTPUT FOURIER TRANSFORM SENSITIVITY VECTORS

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ABSTRACT

We derive a technique for estimating a small number of parameters of a spatially or temporally varying transfer function for which a parametric model is known. Such systems occur in transmission lines fith faults or mismatches, ultrasonic imaging, semiconductor layering and tomography in various applications.

We assume that it is important not only to know that a boundary exists but also to estimate the spatially varying parameters of the medium across the boundary.

The method does not require that the input need be known, but it must be applied to the diverse paths simultaneously, such as with a plane wave hitting a plane surface. The output signals at the various points are time synchronous unless delay is the varying parameter. (In the case of temporal variation the same input must be applied to diverse time windows with corresponding delays.)

The transfer function model may be nonlinear in the parameters, and we may also have to estimate the nominal values around which the parameters are locally varying. The procedure is constructed in the context of estimating at any point in space the parameters of the system by using the eigenstructure of the covariance matrix of vectors whose elements are Fourier transform values of the responses. We require that the available responses in the spatial neighborhood are independent enough to yield a good estimate of the covariance matrix.

It is possible to enumerate and estimate the multiple parameters when the sensitivity vectors (normalized response gradients with respect to the parameters) are not fully correlated. We are constrained to use only frequencies for which output signal to noise ratio (SNR) is reasonably large and for which variations with the parameters are measurable. As an example we examine the case of unknown parameter in a thin layer between 2 semi-infinite layers.

1. INTRODUCTION

There are many applications where spatial or temporal variations of one or more parameters effecting the transmission channel of a signal, in this case a finite energy signal, are of interest. We consider any such channel a transfer function, and the transfer function need not be linear. Example cases are those of a spectrally thin layer such as that in thin films in semiconductors, thin acoustic paths for example in biological tissues, and optical coatings. In all those cases thinness is relative to the wavelengths used to test the material. We show that normalized gradients (sensitivities) of vectors of frequency samples from Fourier transforms of output waveforms, as determined from the models, are linearly related to spectrally normalized vectors from the actual data when the variations are small.

2. OUTPUT FOURIER TRANSFORM VALUES

As shown in figure 1, let the input waveform have Fourier transform values at frequencies f_n arrayed in a column vector $\boldsymbol{u}$ and let the output waveform $\boldsymbol{s}$ have Fourier transform whose respective values are given in the elements of the vector $\boldsymbol{s}(\alpha)$, where α is a vector of unknown transfer function parameters and possibly their products,

$$\boldsymbol{\alpha} = \begin{pmatrix} a_1\ a_2\ \dots\ a_K\ a_1^2\ a_1a_2\ a_1a_3 \\ \dots\ a_{K-1}a_K\ a_K^2\ a_1^3\ a_1^2a_2\ \dots \end{pmatrix}^T \quad (1)$$

The complex transfer function $H(f)$ has amplitude $A(f)$ and phase $\phi(f)$

$$H(f) = A(f)\exp(i\phi(f)) \quad (2)$$

At each frequency f_n, $H(f_n)$ produces an amplitude scaling and a phase shift on each of the elements of $\mathbf{u}$, resulting in $\mathbf{s}(\alpha)$, the signal part of $\mathbf{x} = \mathbf{s}(\alpha)+\mathbf{w}$, where $\mathbf{w}$ is a vector of independent, white, zero-mean Gaussian noises.

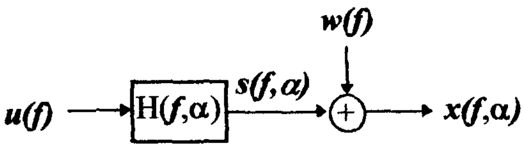

Figure 1. Possibly nonlinear transfer function H(f) with associated parameters α, inputs, outputs and additive noise *w(f)*.

By expanding the noise-free portion of the *reflection* data vectors $\mathbf{x}$ in a Taylor series around their values at the nominal parameters α_0, we have

$$\mathbf{x}-\mathbf{s}(\alpha_0)=\delta_{\mathbf{x}}=\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\delta_\alpha+\mathbf{w}\ , \qquad (3)$$

where the vector δ_α is the deviation from the nominal α_0. The multiple parameter gradient and transformation matrix Γ is defined through

$$\mathbf{D}_{\mathbf{u}}\Gamma=\mathbf{D}_{\mathbf{u}}\left.\frac{\partial H}{\partial \alpha}\right|_{\alpha=\alpha_0}=\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}, \qquad (4)$$

and has elements

$$\left(\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\right)_{n,j}=c_j u_n\left.\frac{\partial^{q_j} H(f_n,\alpha)}{\partial \alpha_j}\right|_{\alpha=\alpha_0}=c_j u_n\frac{\partial^{q_j} H(f_n,\alpha_0)}{\partial \alpha_j}, \qquad (5)$$

where u_n is the incident Fourier transform at f_n, q_j is the sum of the powers of factors of α_j, the j[th] element in α, $\partial\alpha_j$ indicates $\partial\alpha_1^{j_1}\partial\alpha_2^{j_2}\ldots\partial\alpha_K^{j_K}$, and c_j is the associated coefficient of the q_j[th] order derivative in the Taylor expansion. Thus for example the vector element $a_1^2 a_2$ has an associated column of $\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}$ whose elements are

$$\frac{3}{3!}\cdot u_n\frac{\partial^3 H(f_n,\alpha_0)}{\partial^2 a_1\partial a_2}.$$

2. SENSITIVITY VECTORS AND THE SPECTRALLY NORMALIZED DATA COVARIANCE MATRIX

We now define the sensitivity vectors $\mathbf{g}_j$ to be the j[th] column of the sensitivity matrix

$$\mathbf{G}=\mathbf{D}_H^{-1}\left.\frac{\partial H}{\partial \alpha}\right|_{\alpha=\alpha_0}, \qquad (6)$$

wherein the rows of $\left.\frac{\partial H}{\partial \alpha}\right|_{\alpha=\alpha_0}$ have been normalized by the inverse of $\mathbf{D}_H=diag(H(f_n,\alpha_0))$. Thus the nth element of $\mathbf{g}_j$ is

$$\mathbf{g}_{nj}=\frac{c_j}{H(f_n,\alpha_0)}\frac{\partial^{q_j} H(f_n,\alpha_0)}{\partial \alpha_j}. \qquad (7)$$

The vectors $\mathbf{g}_j$ are those that can be matched to the signal subspace given by the covariance matrix of the spectrally normalized data vectors.

Normalization of the vectors $\delta_{\mathbf{x}}$ by the nominal reflected signal transform value gives

$$\begin{aligned}\tilde{\delta}_{\mathbf{x}}&=\mathbf{D}_{\mathbf{s}}^{-1}\delta_{\mathbf{x}}=\mathbf{D}_{\mathbf{s}}^{-1}\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\delta_\alpha+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{w}, \qquad (8)\\&=\tilde{\delta}_{\mathbf{s}}+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{w}\end{aligned}$$

where $\tilde{\delta}_{\mathbf{s}}$ is the normalized deviation of the signal and the data deviation covariance is

$$\begin{aligned}\tilde{\mathbf{C}}_x&=E\{\tilde{\delta}_{\mathbf{x}}\tilde{\delta}_{\mathbf{x}}^H\}=\mathbf{D}_{\mathbf{s}}^{-1}\left(\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\right)E\{\delta_\alpha\delta_\alpha^H\}\\&\left(\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\right)^H\mathbf{D}_{\mathbf{s}}^{-H}+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{N}_{\mathbf{w}}\mathbf{D}_{\mathbf{s}}^{-H} \qquad (9)\\&=\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{R}_{\delta_s}\mathbf{D}_{\mathbf{s}}^{-H}+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{N}_{\mathbf{w}}\mathbf{D}_{\mathbf{s}}^{-H}\end{aligned}$$

where

$$\mathbf{D}_{\mathbf{s}}=diag(u_n H(f_n,\alpha_0)),$$

$$\mathbf{R}_{\delta_s}=\left(\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\right)\mathbf{R}_{\delta_\alpha}\left(\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}\right)^H, \qquad (10)$$

$$\mathbf{N}_{\mathbf{w}}=E\{\mathbf{w}\mathbf{w}^H\}=\sigma^2\mathbf{I},\ \mathbf{R}_{\delta_\alpha}=E\{\delta_\alpha\delta_\alpha^H\}$$

Noting that $\mathbf{D}_{\mathbf{s}}^{-1}\left.\frac{\partial \mathbf{s}}{\partial \alpha}\right|_{\alpha=\alpha_0}=\mathbf{G}=\mathbf{D}_H^{-1}\Gamma$, then (8) can be rewritten

$$\tilde{\delta}_{\mathbf{x}}=\mathbf{G}\delta_\alpha+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{w} \qquad (11)$$

and

$$\begin{aligned}\tilde{\mathbf{C}}_x&=E\{\tilde{\delta}_{\mathbf{x}}\tilde{\delta}_{\mathbf{x}}^H\}\\&=\mathbf{G}E\{\delta_\alpha\delta_\alpha^H\}\mathbf{G}^H+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{N}_{\mathbf{w}}\mathbf{D}_{\mathbf{s}}^{-H}, \qquad (12)\\&=\tilde{\mathbf{R}}_{\delta_s}+\mathbf{D}_{\mathbf{s}}^{-1}\mathbf{N}_{\mathbf{w}}\mathbf{D}_{\mathbf{s}}^{-H}\end{aligned}$$

where $\tilde{\mathbf{R}}_{\delta_s}=\mathbf{G}E\{\delta_\alpha\delta_\alpha^H\}\mathbf{G}^H$ is the spectrally whitened signal covariance matrix. A reasonably good estimate of this can be found by averaging each Fourier transform over all the reflections in a small region of the space where the layer typifies the nominal reflection:

$$\hat{\mathbf{s}} = N^{-1}\sum_{n=1}^{N}\mathbf{x}_n \underset{N\to\infty}{\longrightarrow} E\{\mathbf{x}\} = \mathbf{s} = \mathbf{D}_H\mathbf{u} \qquad (13)$$

The average of the vectors does not necessarily yield a vector that is the correct function of the nominal parameters. However for small deviations it is a good approximation, and approaches the true function if the relationship between parameters and measurements is truly linear.

We will call the eigenvectors associated with the non-zero eigenvalues of $\mathbf{R}_{\delta_s}$ the signal subspace eigenvectors [1], even though they span the space of the normalized *deviations* of the signal vectors rather than the space of the signals themselves. The number r of the independent vectors $\mathbf{g}_j$ which span this spectrally normalized signal space determines the dimension of that subspace and the rank r of the matrix

$$\mathbf{R}_{\delta_\alpha} = E\{\delta_\alpha \delta_\alpha^H\} \qquad (14)$$

Note that the noise covariance matrix has been colored by the normalization in (9). Thus to estimate the signal and noise terms in (8), it is appropriate first to estimate the signal deviation covariance matrix $\mathbf{R}_{\delta_s}$ by estimating and then removing the estimate of $\mathbf{N}_w = \sigma_n^2\mathbf{I}$ from the sample variations covariance matrix of $\mathbf{x}$, then to normalize $\mathbf{R}_{\delta_s}$ by forming $\widetilde{\mathbf{R}}_{\delta_s} = \mathbf{D}_s^{-1}\mathbf{R}_{\delta_s}\mathbf{D}_s^{-1}$.

3. SOLUTION FOR NOMINAL PARAMETERS AND ESTIMATION OF THE PARAMETERS AT EACH REFLECTION POINT

The estimation of the nominal parameters is accomplished by solving for the vectors $\mathbf{g}_j(\alpha_0)$ which span the spectrally normalized signal subspace. One approach uses the MUSIC algorithm [2], and another, correlation, simply looks for the maximum in $\mathbf{g}_j^H\widetilde{\mathbf{R}}_{\delta_s}\mathbf{g}_j$. where a signal subspace estimate of $\widetilde{\mathbf{R}}_{\delta_s}$ may be used.
Once the nominal parameters are known, then both the gradient matrix $\Gamma = \left.\dfrac{\partial H}{\partial \alpha}\right|_{\alpha=\alpha_0}$ and the sensitivity matrix $\mathbf{G} = \mathbf{D}_H^{-1}\Gamma$ are known. We can obtain the Wiener solution of δ_α from (11), where $\widetilde{\delta}_w = \mathbf{D}_s^{-1}\mathbf{w}$ is the normalized noise. Using the orthogonality theorem we find the deviation estimate

$$\begin{aligned}\hat{\delta}_\alpha &= \mathbf{R}_{\delta_\alpha}\mathbf{G}^H\widetilde{\mathbf{C}}_x^{-1}\widetilde{\delta}_x \\ &= \mathbf{R}_{\delta_\alpha}\mathbf{G}^H\left(\mathbf{G}\mathbf{R}_{\delta_\alpha}\mathbf{G}^H + \mathbf{D}_s^{-1}\mathbf{N}_w\mathbf{D}_s^{-H}\right)^{-1}\widetilde{\delta}_x\end{aligned} \qquad (15)$$

The estimates of the parameters themselves is

$$\hat{\alpha} = \alpha_0 + \hat{\delta}_\alpha \qquad (16)$$

4. APPLICATION TO SYNTHETIC DATA

We have produced sensitivity vectors for a suite of acoustic thicknesses 0.001' through 0.150' corresponding to the synthesized model in figure 2. The other nominal parameters for the three media are velocities 6000, 5000, and 6000 ft/sec for layers top to bottom, densities are all unity and angle of test pulse arrival is 0 degrees, orthogonal to the upper boundary. The delay between the two reflections from the upper and lower boundaries in the synthetic data varies linearly from 0 through 50μs. Thus at constant velocity in the middle layer, thickness is varying linearly from 0 through 0.125'. Having these sensitivity functions, we correlate them with the Fourier transform vectors' deviations' covariance matrix, normalized by the mean reflection vector over the range of traces 36 through 40, and obtain figure 3. Both the MUSIC method and the signal subspace correlation method are shown, and these give the same result. The MUSIC method often requires some knowledge of the general location of the "solution" vector; thus the peak near 0.003' should be ignored.

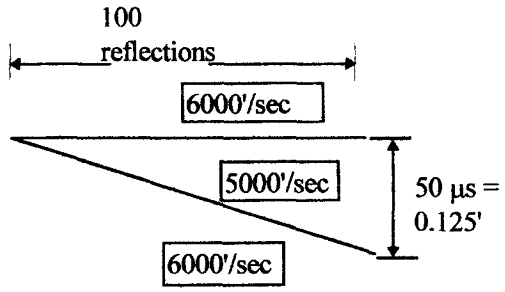

Figure 2. Wedge shaped acoustic object.

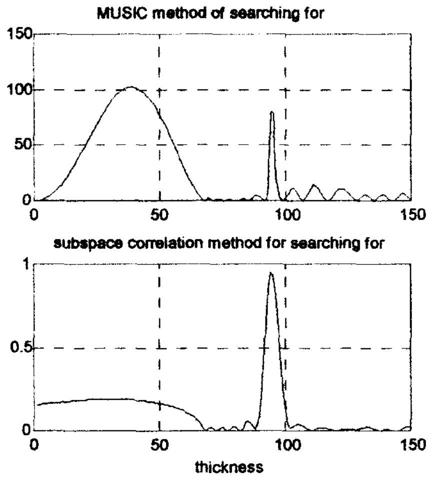

Figure 3. MUSIC and noise covariance correlation methods of search for nominal thickness, traces 36 through 40. (thickness in ft x 10^{-3})

Figure 4 shows the comparison of the chosen sensitivity vector (for thickness equal to 0.094') with the first eigenvector. There is an excellent magnitude match and a good match of the shape of the phase deviation around the most sensitive frequency region near 27,000 Hz..

What we actually want is an estimate of the thickness at each reflection point. Using the gradient vector for the nominal 0.094', each of the trace's Fourier transform vectors' deviations from nominal is processed for deviation of thickness. The results are shown in Figure 5. The real parts of the processing results have been added to the nominal thickness; the true thickness for the model and the imaginary part of the computed deviation are also shown.

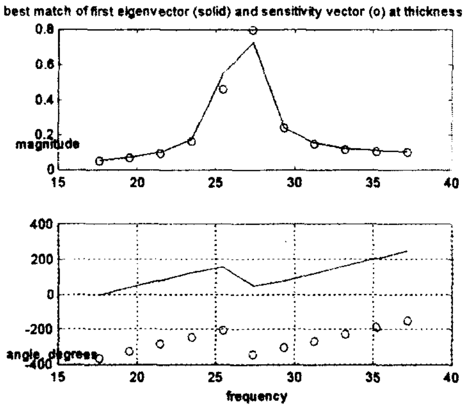

Figure 4. Magnitudes and unwrapped angles of best matching energy-normalized sensitivity vector and first eigenvector, for traces 36 through 40. (Frequency in kilohertz.)

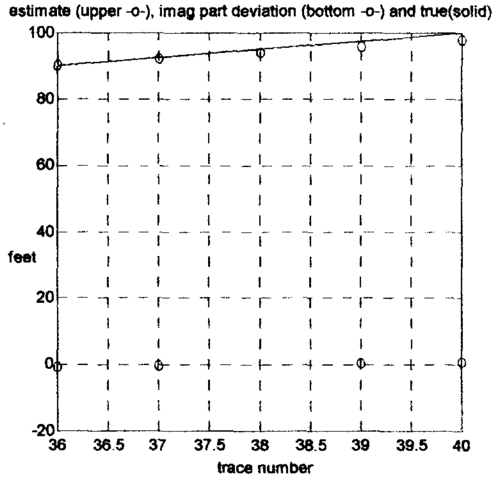

Figure 5. Traces 36 through 40. Real first principal component added to nominal thickness (solid); imaginary part of first principal component (o).

5. CONCLUSIONS

The results of this approach are convincing. In these simulations, we know exactly the velocity of all three layers and have found the unknown thickness. It is expected that if thickness were known, similarly good results would be obtained if a variable velocity were to be determined. The concept has also been extended to multiple variables and nonlinearities. For low orders the scheme should be practical. An outstanding feature of the method is that the incident waveform does not need to be known.

Applications to other parameterized objects is straightforward, and visualization through pseudo 3D imaging of parameter estimates is a typical extension of the results.

REFERENCES

1. L.L. Scharf, *Statistical Signal Processing*, Addison Wesley, Reading, MA, 1991

2. S. Haykin, *Adaptive Filter Theory*, 2nd edition, Prentice Hall, Englewood Cliffs, NJ, 1991.