ON THE EXISTENCE OF UNDESIRABLE GLOBAL MINIMA OF GODARD EQUALIZERS

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ABSTRACT

We consider the problem of global convergence of Godard equalizers in the special case of binary (2-PAM) input signals, when the channel impulse response is complex. We present a class of global minima of all Godard equalizers for this case, which do not correspond to settings free of intersymbol-interference (ISI). The equalizer output corresponding to these global minima appears as a four-point constellation in the complex plane, however it is easily shown that the decomposition in its real and imaginary part provides two ISI-free versions of the transmitted signal. In the case of multi-user constant modulus algorithms, the situation is somewhat more complicated: the real and imaginary parts of each equalizer output after convergence, may correspond to different user signals. These results can be extended to other types of real input signals.

1. INTRODUCTION

The issue of global convergence of the so-called Godard algorithms for blind equalization has been given considerable attention during the last years. When sampled at the symbol rate, the output of a linear communication channel is typically given in the baseband as

$$y(k) = \sum_{l=0}^{L-1} h_l a(k-l) + n(k) \tag{1}$$

where $\{a(k)\}$ is the transmitted symbol sequence, $\{h_l\}$ the channel impulse response (assumed to be time-invariant) and $\{n(k)\}$ is the additive noise at the channel output, all sampled at the symbol rate. L is the channel length, measured in symbol periods. When $\{h(k)\}$ is not a Dirac impulse (up to a multiplicative factor), the signal $y(k)$ is corrupted by intersymbol interference (ISI).

In his original paper [1] Godard proposed the following cost functions for blind equalization (in order to reduce the ISI) of QAM signals:

$$J_p^{God}(W) = \frac{1}{2p} E\left(|z(k)|^p - r_p\right)^2 \quad , \; p = 1, 2, \dots \tag{2}$$

where E denotes statistical expectation, and $z(k)$ is the sampled equalizer output at time instant k given by

$$z(k) = \sum_{l=0}^{M-1} w_l(k) y(k-l) = Y^T(k) W(k) \tag{3}$$

where $W(k) = [w_1(k) \cdots w_{M-1}(k)]^T$ and $Y(k) = [y(k) \cdots y(k-M+1)]^T$ are the equalizer setting and vector channel output at time instant k, respectively. r_p is a constant scalar called *dispersion constant* and defined as:

$$r_p = \frac{E|a(k)|^{2p}}{E|a(k)|^p} \tag{4}$$

The minimization of the above cost function with respect to the equalizer vector W yields the following class of stochastic gradient algorithms, which are known as the Godard algorithms:

$$W(k+1) = W(k) + \mu Y^*(k) z(k) |z(k)|^{p-2} (r_p - |z(k)|^p) \tag{5}$$

where * denotes complex conjugate and ^{T} matrix transpose. A choice for p in (5) equal to 1 or 2 corresponds to the popular constant modulus (CM) algorithms CMA 1-2 and CMA 2-2, respectively [2].

The convergence properties of the Godard algorithms have been extensively investigated in the literature. Godard showed in [1] that the global minima of the cost functions correspond to ideal zero forcing (in the absence of noise and for infinite equalizer lengths) equalizer settings and predicted the existence of local stationary points that correspond to undesired solutions. In order to obtain these results, two assumptions on the input signal $\{a(k)\}$ where made, given in the following two equations:

$$E\left(a^2(k)\right) = 0 \tag{6}$$

$$E(|a(k)|^4) < 2\left(E^2(|a(k)|^2)\right) \tag{7}$$

Eq. (6) imposes a *symmetry* property on the transmitted constellation, whereas Eq. (7) assumes some *compactness* of the constellation. In [3], Sato has analyzed the Sato algorithm (which coincides with the CMA 1-2 in the case of real signals) and found that for multi-level PAM signals the algorithm will converge to an ideal setting if the channel eye is initially open. Benveniste et al [4] extended this result by proving that even for an initially closed channel eye the algorithm will converge to an ideal setting (assuming again an infinite equalizer length and no additive noise), provided that the input signal has a continuous sub-Gaussian distribution. Shalvi and Weinstein [5] provided a strong result for the CMA 2-2: its global convergence property for any sub-Gaussian and symmetric input distribution, provided again that the equalizer has infinite length and no additive noise is present. The symmetry property assumed is the same as (6), whereas sub-Gaussianity is guaranteed if the following condition holds:

$$K(a) < 0 \tag{8}$$

where $K(a)$ is the *kurtosis* of the input, defined as

$$K(a) = E\left(|a(k)|^4\right) - 2E^2\left(|a(k)|^2\right) - \left|E\left(a^2(k)\right)\right|^2 \tag{9}$$

Notice that when (6) is satisfied, then (8) takes the form

$$\overline{K}(a) = \frac{E\left(|a(k)|^4\right)}{E^2\left(|a(k)|^2\right)} < 2 \qquad (10)$$

where $\overline{K}(a)$ is the so-called ***normalized kurtosis*** of a. Notice that (10) is equivalent to (7).

Ding et al [6], [7], [8] brought up the issue of local minima of Godard equalizers when the equalizer has finite length and showed that the initialization of the algorithm plays an important role in the convergence to local or global minima. More recently, Mayrargue [9], Li and Ding [10] and Fijalkow et al [11] showed that in the case of SIMO (single-input-multiple-output) channels (corresponding to fractionally spaced equalization obtained by oversampling or to antenna array reception) finite length Godard equalizers are globally convergent to ideal equalizer settings in the noiseless case if a certain zeros-and-length condition holds for the multichannel. Again the results assume the two conditions (6) and (8).

In this paper we consider a special case when the symmetry condition (6) is not met, and more specifically the case of real PAM input signals when the receiver is complex (implying complex transmission channel, equalizer, and additive noise). The impact of these assumptions on the performance of both single-user and multi-user CM-type algorithms gives several interesting results that we present in the sequel.

2. SINGLE-USER ALGORITHMS

We first assume that the transmitted input signal is a zero-mean 2-PAM (BPSK) signal, which takes on the values +1 or −1 with equal probabilities:

$$\Pr(a(k)=1) = \Pr(a(k)=-1) = \frac{1}{2}, \qquad \forall\, k \qquad (11)$$

Notice that this signal is sub-Gaussian $K(a) = -2 < 0$, however it is not symmetric in the complex plan: $E(a^2(k)) = 1 \neq 0$.

We also assume that the transmission channel is complex. This will be the case when the receiver is complex (i.e. processes the signals in both in-phase and quadrature) which corresponds to complex baseband processing, despite the fact that the transmitted signal is real (see e.g. [12] for a similar scenario).

We also assume that the equalizer W is a complex vector that is updated through any of the Godard algorithms described by (5). We now consider an equalizer output of the following form:

$$z(k) = e^{j\theta}\left(cos\phi\, a(k-l_1) + jsin\phi\, a(k-l_2)\right) \qquad (12)$$

where $j = \sqrt{-1}$, l_1, l_2 are arbitrary integers and θ, ϕ are random phases ($\theta, \phi \in [0, 2\pi)$). For the input signal described by (11) the dispersion constant given by (4) is equal to 1 for all values of p:

$$r_p = 1\,, \qquad \forall\, p \qquad (13)$$

The Godard cost function (2) in the case of an equalizer output of the form in (12) will therefore equal:

$$J_p^{God}(W) = 0\,, \qquad \forall\, p \qquad (14)$$

According to (14) all the solutions of the form (12), corresponding to a channel-equalizer cascade impulse response g ($g = h \star w$, where $\star$ denotes convolution) of the following form

$$g = e^{j\theta}\left(\cdots 0 \cdots 0\, cos\phi\, 0 \cdots 0\, jsin\phi\, 0 \cdots 0 \cdots\right) \qquad (15)$$

are global minima (since $J_p^{God}(g)$ is a nonnegative cost function) of all the Godard cost functions $J_p^{God}(g)$. Note that the existence of more than one non-zero terms in g indicates the presence of intersymbol interference. This result is outlined in the following theorem:

Theorem I: In the case of a 2-PAM input signal and a complex channel-equalizer cascade g, the Godard cost functions of all orders p admit the settings of the form (15) as global minima.

The following points are worth to be discussed about the implications of the above theorem.

Discussion

When $l_1 \neq l_2$, then the settings of the form (15) are not ISI-free, for example the closed-eye measured defined as

$$\rho = \frac{\sum_i |g_i| - \max_i(|g_i|)}{\max_i(|g_i|)} \qquad (16)$$

will equal 1 if $\phi = \frac{\pi}{4}$ (0 is the ISI-free value for ρ). When $l_1 = l_2$, the setting g takes the form

$$g = \left(\cdots 0\, e^{j\phi'}\, 0 \cdots\right) \qquad (17)$$

($\phi' = \phi + \theta$) corresponding to the well-known optimal stationary points (ϕ' now represents a phase ambiguity which is inherent in blind equalization and causes an arbitrary rotation of the constellation).

Even though, strictly speaking, the stationary points of the form (15) contain residual ISI after the equalizer convergence, they are not truly undesirable. One may notice that if the phase ambiguity $e^{j\theta}$ is removed, and one separates the real and imaginary parts of g, the resulting settings are both optimal and provide ISI-free settings that reveal the transmitted 2-PAM sequences at different delays.

The existence of the global minima points (15) can be interpreted as follows: the transmitted 2-PAM constellation will not appear after equalization as an arbitrarily rotated 2-PAM constellation, as one might expect, but rather as an arbitrarily rotated 4-point constellation! Each of the two pairs of this constellation will carry however, as already mentioned, the ISI-free transmitted data.

In the light of Theorem I, it is clear that the results obtained in [10], [9], [11], can be extended to include this particular case for which $E(a^2) = 0$ is not satisfied. Clearly, in the 2-PAM case, the polyphase CMA 2-2 algorithm (operating on an over-sampled received signal and/or using the outputs of an antenna array) will admit (in the absence of noise and common roots between all the different channel phase polynomials) the settings of the form (15) as its global minima.

It is also worth noting that Theorem I applies to Godard functions of all orders p (see (2)). Moreover, similar results can be obtained for decision-directed cost functions. Finally, it can be also shown that similar results are obtained for multi-level PAM inputs.

3. MULTI-USER ALGORITHMS

CM algorithms have also been proposed in a number of cases for either separating or finding one out of several co-channel users. The emphasis is on the removal of the inter-user interference (IUI). In the absence of significant channel delay spread, the CM beamformer [13], [14] has been shown [15] to have global solutions that correspond to IUI-free output signals, provided that the corresponding user signal is symmetric sub-Gaussian (equations (6) and (7) are jointly satisfied). However, it may not be possible with this scheme to recover all the user signals, or even predict which signal will be extracted. Two alternative approaches to combat this problem are the multi-stage CMA [16] and the MIMO-CMA proposed in [17]. In the presence of delay spread, the multi-user CMA proposed in [18] is able of combatting jointly both the ISI and the IUI of the channel.

We now focus on the latter approach [18], since its cost function can be seen as a generalization of the cost functions for the algorithms proposed in [13] [14] and [17]. Assuming a p-input-m-output MIMO linear channel and a m-input-p-output MIMO linear equalizer, we denote by $a_j(k)$, $j \in \{1,\ldots,p\}$ the j^{th} transmitted signal, $n_i(k)$ the additive noise sample at channel output i, $y_i(k)$, $i \in \{1,\ldots,m\}$ the noisy received signal at channel output i and $z_j(k), j \in \{1,\ldots,p\}$ the j^{th} equalizer output, all at time instant k. We also denote by H_{ij}, $i \in \{1,\ldots,m\}$, $j \in \{1\ldots p\}$ a $L_j \times 1$ vector representing the linear channel from input j to channel output i and by W_{ij} a $N \times 1$ vector that contains the coefficients of the linear filter relating the i^{th} received signal to the j^{th} output (we have assumed the channel H_{ij} to have L_j non-zero discrete samples and each equalizer W_{ij} to have N coefficients). The signal model containing both ISI and IUI is given by

$$y_i(k) = \sum_{j=1}^{p} H_{ij}^T A_j(k) + n_i(k)\ ,\ i \in \{1,\ldots,m\} \tag{18}$$

and the equalizer outputs by

$$z_j(k) \ = \ \sum_{i=1}^{m} W_{ij}^T Y_i(k)\ ,\ j \in \{1,\ldots,p\} \tag{19}$$

where

$$A_j(k) = [a_j(k)\cdots a_j(k-N_j+1)]^T \tag{20}$$

and

$$Y_i(k) = [y_i(k)\cdots y_i(k-L+1)]^T \tag{21}$$

The MIMO-CMA cost function is given by

$$J_2(\mathbf{W}) \ = \ E\sum_{j=1}^{p}\left(|z_j|^2-1\right)^2 \ + 2\sum_{l,n=1\ ;\ l\neq n}^{p}\ \sum_{\delta=\delta_1}^{\delta_2}|r_{ln}(\delta)|^2 \tag{22}$$

where $\mathbf{W}$ is the $Nm \times p$ equalizer matrix defined as

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_{11} & \cdots & \mathbf{w}_{1p} \\ \vdots & \vdots & \vdots \\ \mathbf{w}_{m1} & \cdots & \mathbf{w}_{mp} \end{bmatrix} \tag{23}$$

with

$$\mathbf{w}_{ij} = [W_{ij}(0)\ \cdots\ W_{ij}(N-1)]^T \tag{24}$$

$r_{ln}(\delta)$ is the cross-correlation function between users l and n defined as

$$r_{ln}(\delta) = E\left(z_l(k)z_n^*(k-\delta)\right) \tag{25}$$

If each signal $a_j(k)$ satisfies both equations (6) and (7), it is possible to show [18] that the global minima of (22) correspond to a vector equalizer output that contains possibly-shifted and rotated by an arbitrary scalar versions of *all* the different input signals (provided that some channel length-and-zeros conditions hold). If the cross-correlation terms were omitted in (22), each equalizer output would be updated by a CMA 2-2 algorithm

$$W_j(k+1) = W_j(k) - \mu\mathbf{Y}^*(k)(z_j(k)(|z_j(k)|^2-1)) \tag{26}$$

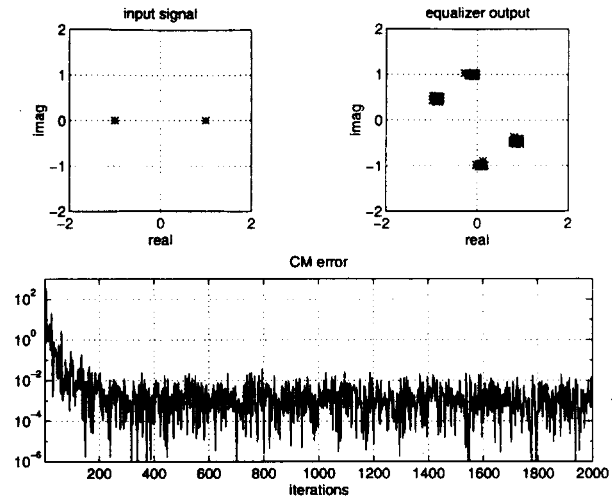

Figure 1. The 2-PAM case

in which case it is found that each output $z_j(k)$ will be an interference-free version of *any* of the input signals – this coincides with the result in [15] –.

However, when the input is 2-PAM, it is easy to show that (since each signal $a_j(k)$ satisfies (7)), the cost function (22) admits as global minima the settings that correspond to equalizer outputs of the following form:

$$z_j(k) \ = \ e^{\sqrt{-1}\theta}\left(cos\phi\, a_{j1}(k-l_1) \ + \ \sqrt{-1}sin\phi\, a_{j2}(k-l_2)\right) \tag{27}$$

where $j1$ and $j2$ can take *any* value in $\{1,\ldots,p\}$. Therefore, the output of each filter will not only appear as a four-point constellation, but moreover, when decomposed in its real and imaginary parts, it will reveal interference-free versions of possibly *different* input signals. This performance will appear in all the algorithms proposed in [13], [14], [17] and [18], the difference being again that in the first case some of the signals may not be found, whereas in the other two cases the presence of the cross-correlation terms will result in finding all the signals.

Therefore, in the multi-user case, the demodulation of the CM algorithm outputs in the 2-PAM/complex channel case is somewhat more complicated: the different user signals can be found as the real or imaginary parts of different equalizer outputs. Similar results apply to the analytical CMA presented in [19].

4. COMPUTER SIMULATION RESULTS

The following simple computer simulation result is presented in support of the validity of Theorem I. We first consider the case of a 2-PAM signal transmitted through a channel with complex impulse response whose coefficients were randomly chosen as

$$\begin{aligned} h[1] &= [\ -0.72+1.52j \quad -0.56+1.62j \quad +0.51-0.76j\ ] \\ h[2] &= [\ -0.86+0.52j \quad +1.22-1.86j \quad +0.72+0.53j\ ] \end{aligned}$$

where $h[1]$ and $h[2]$ represent the odd end even parts of the impulse response, respectively (we consider fractionally-spaced equalization with an over-sampling factor of 2).

We run the CMA 2-2 algorithm (with a stepsize optimized for maximal convergence speed) using 9 taps for each of the two phases of the fractional equalizer (each one is initialized with a center-spike pattern). Additive noise of SNR=40 dB is added to the received signal. As can be seen in Figure 1, the algorithm converges quickly to a 4-point pattern:

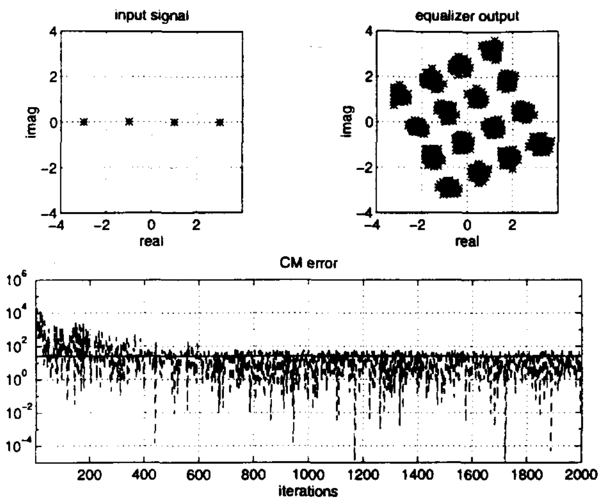

Figure 2. The 4-PAM case

it is found that each of the two pairs of this constellation corresponds to the transmitted 2-PAM signal at 2 different delays. This verifies the validity of Theorem I and the prediction of the expected 4-point pattern. A similar behavior can be observed for the 4-PAM case, as shown in Figure 2. In this case the transmitted input belongs with equal probabilities to the alphabet $\{-3, -1, 1, 3\}$, and the same complex channel as in the previous experiment is used. Notice how after convergence the 4-point input has resulted to a 16-point pattern equalizer output. Other simulation results have shown the validity of (27) for the multi-user 2-PAM case.

5. CONCLUSIONS

We have identified a new class of global minima of Godard-type equalizers in the case of 2-PAM input signals and complex channel-equalizer cascades. These sets of global minima have two non-zero elements in the channel-equalizer impulse response, indicating the presence of ISI and resulting in a 4-point output constellation. However the orthogonality between the two non-zero elements results in a separability of the obtained signal: when decomposed in its real and imaginary parts, these will be free of ISI. Similar results hold for other non-symmetric input signals, as well as for multi-user CM algorithms. Apart from treating theoretically a case that seems to have been neglected to date (since it corresponds to non-symmetrical input signals) these results indicate how demodulation must be performed when both single-user and multi-user CM algorithms are used for blind equalization.

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