A NEW DISCRETE FRACTIONAL FOURIER TRANSFORM BASED ON CONSTRAINED EIGENDECOMPOSITION OF DFT MATRIX BY LARGRANGE MULTIPLIER METHOD

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ABSTRACT

This paper is concerned with the definition of the discrete fractional Fourier transform (DFRFT). First, an eigendecomposition of the discrete Fourier transform (DFT) matrix is derived by sampling the Hermite Gauss functions which are eigenfunctions of the continuous Fourier transform and by performing a novel error removal procedure. Then, the result of the eigendecomposition of the DFT matrix is used to define a new DFRFT. Finally, a numerical example is illustrated to demonstrate the proposed DFRFT is a better approximation to the continuous fractional Fourier transform than the conventional defined DFRFT.

1. INTRODUCTION

In recent years, many researchers have paid attention to the investigation of a new signal processing tool called fractional Fourier transform (FRFT). This transform has found many applications in the solution of differential equation, quantum mechanics and quantum optics, and optical systems and optical signal processing, swept-frequency filter, time-variant filtering and multiplexing, pattern recognition, and study of time-frequency distribution [1]. Besides, the FRFT has been proved to relate to other signal analysis tools, such as Wigner distribution, neural network, wavelet transform and various chirp-related operations [2]. Several useful properties of FRFT are currently under study in signal processing community [3].

So far, many methods for implementing FRFT has been developed. However, most of them are to utilize the optical instruments or numerical integration. Because the FRFT is a potentially useful tool for signal processing, the direct computation of FRFT in digital computer has become an important issue. Basically, the computation of the discrete fractional Fourier transform (DFRFT) needs to obey additivity property and similarity condition. The additivity property means that application of the transform with angular parameter α followed by an application of the transform with angular parameter β is equivalent to the application of the transform with angular parameter $\alpha+\beta$. The similarity condition means that the transform results of DFRFT must be similar to those of the continuous FRFT. In [4], a method for digital computing FRFT was proposed, but their method does not obey the additivity property and the signal can not be recovered from its transform results. In [5], another DFRFT is defined, but this definition does not provide the similar transform results as those of continuous case. The purpose of this paper is to present a new DFRFT which obey additivity property and similarity condition simultaneously.

2. EIGENDECOMPOSITION OF THE DFT MATRIX

2.1 The eigenvalues and eigenvectors of DFT matrix

Now, we briefly review the properties of the eigenvalues and eigenvectors of the DFT matrix $\mathbf{F}$ whose elements defined by

$$\mathbf{F}_{nk} = \frac{1}{\sqrt{N}}\left(\cos(\frac{2\pi kn}{N}) - j\sin(\frac{2\pi kn}{N})\right) \qquad 0 \le n,k \le N-1 \tag{1}$$

From the results in [6][7], the properties can be summarized as the following two facts:

Fact 1 The eigenvalues of $\mathbf{F}$ are $\{1,-1,j,-j\}$ and its multiplicities are listed below:

N	Mul. of 1	Mul. of -1	Mul. of -j	Mul. of j
4m	m+1	m	m	m-1
4m+1	m+1	m	m	m
4m+2	m+1	m+1	m	m
4m+3	m+1	m+1	m+1	m

Fact 2 Let $\omega = \frac{2\pi}{N}$ and matrix $\mathbf{S}$ be

$$\mathbf{S} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2\cos(\omega) & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 2\cos((N-1)\omega) \end{bmatrix} \tag{2}$$

then it can be shown that $\mathbf{FS} = \mathbf{SF}$.

Because matrix $\mathbf{S}$, with distinct eigenvalues, commutes with $\mathbf{F}$, the eigenvectors of $\mathbf{S}$ will be the desired set of eigenvectors of $\mathbf{F}$. Note that $\mathbf{S}$ is a real and symmetric matrix, so its eigenvectors will be real and orthogonal. Although Fact 2 can help us to find a real orthogonal eigenvector set of the matrix $\mathbf{F}$, this solution is not unique because any linear combination of the eigenvetors which correspond to the same eigenvalue is also an eigenvector. Thus, there exist infinite eigendecomposition forms of the DFT matrix. If we use the eigendecomposition of the DFT matrix $\mathbf{F}$ to define the discrete fractional Fourier transform (DFRFT), then we have infinite choice. However, under the condition that transform results of DFRFT needs to be similar to those of continuous FRFT, the eigendecomposition of DFT matrix must be found trickly. In the following, we will derive an eigendecomposition form by sampling the Hermite Gauss functons which are the eigenfunctions of the continuous Fourier transform and by performing a novel error removal propcedure. Using the proposed decomposition to

define DFRFT, the transform results will obey simialarity condition.

2.2 An eigendecomposition of DFT matrix

The usual continouos Fourier transform pair is defined as

$$\mathbf{X}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{X}(\omega) e^{j\omega t} d\omega$$

It can be shown that the eigenfunctions of the Fourier transformation operator are Hermite Gauss function $H_m(t)e^{-\frac{t^2}{2}}$, where $H_m(t)$ are the Hermite polynomials of order m. We thus have

$$(-j)^m H_m(\omega) e^{\frac{-\omega^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_m(t) e^{\frac{-t^2}{2}} e^{-j\omega t} dt \quad (3)$$

Now, we will use this equation to derive an approximate eigendecomposition of the DFT matrix. Our derivation is mainly based on the following two facts:

Fact 3: If the sequence $g_m(n)$ is obtained by sampling the Hermite Gauss function $H_m(t)e^{\frac{-t^2}{2}}$ with sampling interval $T = \sqrt{\frac{2\pi}{N}}$, i.e.,

$$g_m(n) = H_m(nT) e^{\frac{-(nT)^2}{2}} \quad (4)$$

then it can be shown that

$$(-j)^m g_m(k) \approx \sqrt{\frac{1}{N}} \sum_{n=\frac{-N}{2}}^{\frac{N}{2}-1} g_m(n) e^{-j\frac{2\pi kn}{N}} \quad (5)$$

for sufficiently large N.

Because the degree of Hermite polynomial $H_m(t)$ is m, the decay rate of the Hermite Gauss function $H_m(t)e^{\frac{-t^2}{2}}$ is proportional to $t^m e^{-t^2}$ for sufficiently large t. And, the larger order m is, the slower decay rate Hermite Gauss function has. Thus, when order m becomes large, the approximation in eq(5) become worse.

Fact 4: If the sequence $\bar{g}_m(n)$ defined in the range $[0, N-1]$ is obtained by shifting Hermite Gauss samples $g_m(n)$ defined in the range $[\frac{-N}{2}, \frac{N}{2}-1]$ in the following way:

$$\bar{g}_m(n) = \begin{cases} g_m(n) & \text{for } 0 \le n \le \frac{N}{2}-1 \\ g_m(n-N) & \text{for } \frac{N}{2} \le n \le N-1 \end{cases} \quad (6)$$

then it can be shown that the DFT of the $\bar{g}_m(n)$ can be approximated by $(-j)^m \bar{g}_m(k)$, i.e.,

$$(-j)^m \bar{g}_m(k) \approx \sqrt{\frac{1}{N}} \sum_{n=0}^{N-1} \bar{g}_m(n) e^{-j\frac{2\pi kn}{N}} \quad (7)$$

for sufficiently large N.

From the Fact 4, it is clear that $\bar{g}_m(n)$ are the approximate eigenfunctions of the discrete Fourier transform. Because the Hermite Gauss functions are orthogonal each other for different orders, the sequences $\bar{g}_{m1}(n)$ and $\bar{g}_{m2}(n)$ are approximately orthogonal for $m1 \neq m2$, i.e.,

$$\sum_{n=0}^{N-1} \bar{g}_{m1}(n) \bar{g}_{m2}(n) \approx 0 \quad (8)$$

Let us define the vectors $\mathbf{v}_m$ as follows:

$$\mathbf{v}_m = [\bar{g}_m(0) \;\; \bar{g}_m(1) \cdots \bar{g}_m(N-1)]^t \quad (9)$$

then eq(7) means that

$$(-j)^m \bar{\mathbf{v}}_m \approx \mathbf{F} \bar{\mathbf{v}}_m \quad (10)$$

where $\bar{\mathbf{v}}_m = \frac{\mathbf{v}_m}{\|\mathbf{v}_m\|}$ is normalized version of the vector $\mathbf{v}_m$. Thus, $\bar{\mathbf{v}}_m$ is an approximate eigenvector of the DFT matrix $\mathbf{F}$ corresponding to the eigenvalue $(-j)^m$. Although the approximate expression in eq(10) is valid for any order m, the DFT matrix $\mathbf{F}$ with size $N \times N$ only has N eigenvectors whose eigenvalues need to satisfy the multiplicity property in Fact 1. Thus, we are required to select N orders denoted by the set $\Psi = \{m_1, m_2, \cdots, m_N\}$ $(m_1 < m_2 < \cdots < m_N)$ to construct an eigendecomposition of the matrix $\mathbf{F}$. Two rules of the selection in this paper is listed as follows:

(1) The set $\{(-j)^{m_1}, (-j)^{m_2}, \cdots, (-j)^{m_N}\}$ formed by eigenvalues must satisfy the multiplicity property in Fact 1.

(2) The approximation error $\|(-j)^{m_i} \bar{\mathbf{v}}_{m_i} - \mathbf{F}\bar{\mathbf{v}}_{m_i}\|$ must be less than the error $\|(-j)^m \bar{\mathbf{v}}_m - \mathbf{F}\bar{\mathbf{v}}_m\|$ if m is not in the set Ψ.

Because the approximation error $\|(-j)^m \bar{\mathbf{v}}_m - \mathbf{F}\bar{\mathbf{v}}_m\|$ becomes large when order m increases, a suitable choice of set Ψ which obeys two rules is described in the following table:

N	$\Psi = \{m_1, \cdots, m_N\}$
4n	$0, 1, 2, \cdots, 4n-2, 4n$
4n+1	$0, 1, 2, \cdots, 4n-1, 4n$
4n+2	$0, 1, 2, \cdots, 4n, 4n+2$
4n+3	$0, 1, 2, \cdots, 4n+1, 4n+2$

Based on this choice, an approximate eigendecomposition of the DFT matrix $\mathbf{F}$ is given by

$$\mathbf{F} \approx \sum_{i=1}^{N} (-j)^{m_i} \bar{\mathbf{v}}_{m_i} \bar{\mathbf{v}}_{m_i}^t \quad (11)$$

In order to remove the error in this decomposition, an eigenvector calibration procedure is developed as follows. Assumed that the eigenvector set $\{\bar{\mathbf{v}}_{m_1}, \bar{\mathbf{v}}_{m_2}, \cdots, \bar{\mathbf{v}}_{m_N}\}$ will be corrected into the eigenvector set $\{\mathbf{u}_{m_1}, \mathbf{u}_{m_2}, \cdots, \mathbf{u}_{m_N}\}$ and the vectors from $\mathbf{u}_{m_1}$ to $\mathbf{u}_{m_{k-1}}$ have been obtained. Then, the eigenvector $\mathbf{u}_{m_k}$ is found by minimizing the squared error $(\mathbf{u}_{m_k} - \bar{\mathbf{v}}_{m_k})^2$ subjected to two prescribed constraints which are the eigenvector constraint $\mathbf{F}\mathbf{u}_{m_k} = (-j)^{m_k}\mathbf{u}_{m_k}$ and the orthogonal constraint $\mathbf{u}_i^t \mathbf{u}_{m_k} = 0$ for $i = m_1, \cdots, m_{k-1}$. After some maniputation, two constraints can be rewritten as matrix form below:

$$\mathbf{C}_{m_{k-1}} \mathbf{u}_{m_k} = \mathbf{0} \quad (12)$$

where the matrix $\mathbf{C}_{m_{k-1}}$ is given by

$$\mathbf{C}_{m_{k-1}} = \begin{bmatrix} Real(\mathbf{F} - (-j)^{m_k}\mathbf{I}) \\ Img(\mathbf{F} - (-j)^{m_k}\mathbf{I}) \\ \mathbf{u}_{m_1}^t \\ \vdots \\ \mathbf{u}_{m_{k-1}}^t \end{bmatrix} \quad (13)$$

The notation *Real*(.) and *Img*(.) denote the real part and imaginary part of a complex matrix, and I is identity martix. Using the QR decomposition, the matrix $\mathbf{C}_{m_{k-1}}$ can be rewritten as

$$\mathbf{C}_{m_{k-1}} = \mathbf{Q}_{m_{k-1}} \begin{bmatrix} \mathbf{R}_{m_{k-1}} \\ \mathbf{0} \end{bmatrix} \tag{14}$$

Substitute eq(14) into eq(12), the eq(12) reduces to

$$\mathbf{R}_{m_{k-1}} \mathbf{u}_{m_k} = \mathbf{0} \tag{15}$$

If the rank of matrix $\mathbf{C}_{m_{k-1}}$ is equal to r, the size of the matrix $\mathbf{R}_{m_{k-1}}$ is $r \times N$. Now, using the well-known Largrange multiplier method, the solution of this constrained optimization problem is given by

$$\mathbf{u}_{m_k} = \left(\mathbf{I} - \mathbf{R}^t_{m_{k-1}} (\mathbf{R}_{m_{k-1}} \mathbf{R}^t_{m_{k-1}})^{-1} \mathbf{R}_{m_{k-1}}\right) \bar{\mathbf{v}}_{m_k} \tag{16}$$

Finally, the entire eigenvector calibration procedure is summarized as follows: Given DFT matrix $\mathbf{F}$ and the approximate eigenvector set $\{\bar{\mathbf{v}}_{m_1}, \bar{\mathbf{v}}_{m_2}, \cdots, \bar{\mathbf{v}}_{m_N}\}$ we take the following steps to compute the exact eigenvector set $\{\mathbf{u}_{m_1}, \mathbf{u}_{m_2}, \cdots, \mathbf{u}_{m_N}\}$.

Step 1: Let matrix $\mathbf{C}_{m_1}$ be $[\mathbf{I} - Real(\mathbf{F})^t, Img(\mathbf{F})^t]^t$ and use eq(16) to find the vector $\mathbf{u}_{m_1}$. Note that we normalize $\mathbf{u}_{m_1}$ to unit norm. Set $k = 2$.

Step 2: Perform the following two computations:

(1) Use the eq(13)(14) to compute the matrix $\mathbf{R}_{m_k}$.

(2) Use the eq(16) to calculate the vector $\mathbf{u}_{m_k}$ and normalize it to unit norm.

Step 3: Let $k = k + 1$. If $k > N$, stop the procedure. Otherwise go to *Step 2*.

After this calibration, the exact eigendecomposition of the DFT matrix $\mathbf{F}$ is given by

$$\mathbf{F} = \sum_{i=1}^{N} (-j)^{m_i} \mathbf{u}_{m_i} \mathbf{u}^t_{m_i} \tag{17}$$

The unique feature of this eigendecomposition is that the shape of the eigenvector is similar to the shape of the Hermite Gauss functions which is the eigenfunction of the continuous Fourier transform. In the next section, we will use this decomposition to define a discrete fractional Fourier transform.

3. NEW DEFINITION OF DFRFT

The DFRFT of the data vector $\mathbf{x}$ is defined by

$$\Re^{\alpha}[\mathbf{x}] = \mathbf{F}^{\frac{2\alpha}{\pi}} \mathbf{x}$$

Since $\frac{2\alpha}{\pi}$th power of the DFT matrix $\mathbf{F}$ can be calculated from its eigendecomposition by taking the $\frac{2\alpha}{\pi}$th power for its eigenvalues, the matrix $\mathbf{F}^{\frac{2\alpha}{\pi}}$ is given by

$$\mathbf{F}^{\frac{2\alpha}{\pi}} = \sum_{i=1}^{N} (-j)^{m_i \frac{2\alpha}{\pi}} \mathbf{u}_{m_i} \mathbf{u}^t_{m_i} \tag{18}$$

Because $(-j)^{m_i \frac{2\alpha}{\pi}} = e^{-jm_i\alpha}$, the eigenvalues of the new transform matrix $\mathbf{F}^{\frac{2\alpha}{\pi}}$ are consistent with those of the continuous FRFT. Moreover, the eigenvectors $\mathbf{u}_{m_i}$ are obtained by sampling Hermite Gauss functions with an error removal procedure, so the eigenvectors of new DFRFT are similar to those of the continuous FRFT. Due to these two agreements, the transform result of our DFRFT will be similar to that of continuous FRFT. In order to demonstrate the advantage of our DFRFT, we consider the FRFT of impulse function $\delta(t)$. The continuous FRFT of this special signal has the closed form formula given by

$$\mathbf{F}^{\alpha}[\delta(t)] = \sqrt{\frac{1 - j\cot(\alpha)}{2\pi}} e^{j\frac{u^2}{2}\cot(\alpha)} \tag{19}$$

Fig.1 shows the continuous FRFT of the impulse signal for various angles α. For comparision, we examine the DFRFT of the unit sample function defined by

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Fig.2 shows the trasnform result of the DFRFT defined by Santhanam and McClellan [5], and Fig.3 depicts the result of our DFRFT for $N = 36$. It is clear our result is more similar to that of continuous case than the result of the conventional DFRFT.

4. CONCLUSIONS

In this paper, a new definition of the discrete fractional Fourier transform (DFRFT) based on an eigendecomposition of DFT matrix has been presented. The eigencomposition of the DFT matrix is derived by sampling the Hermite Gauss functions which are eigenfunctions of the continuous fractional Fourier transform and by performing a novel error removal procedure. A numerical numerical example is illustrated to demonstrate the proposed DFRFT is a better approximation to the continuous fractional Fourier transform than the conventional defined DFRFT. However, the complexity for implementing DFRFT is $O(N^2)$ which is same as that of DFT. Thus, it is interesting to develop a fast algorithm to compute DFRFT.

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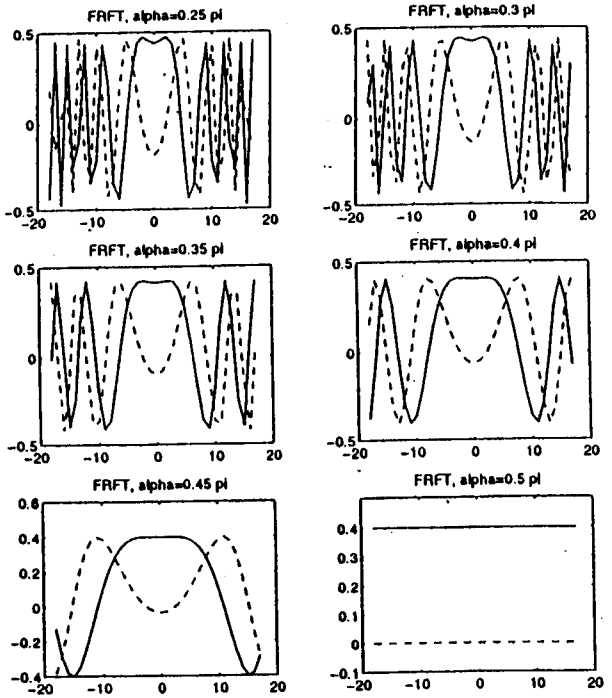

Figure 1 The continuous FRFT of the impulse signal $\delta(t)$.

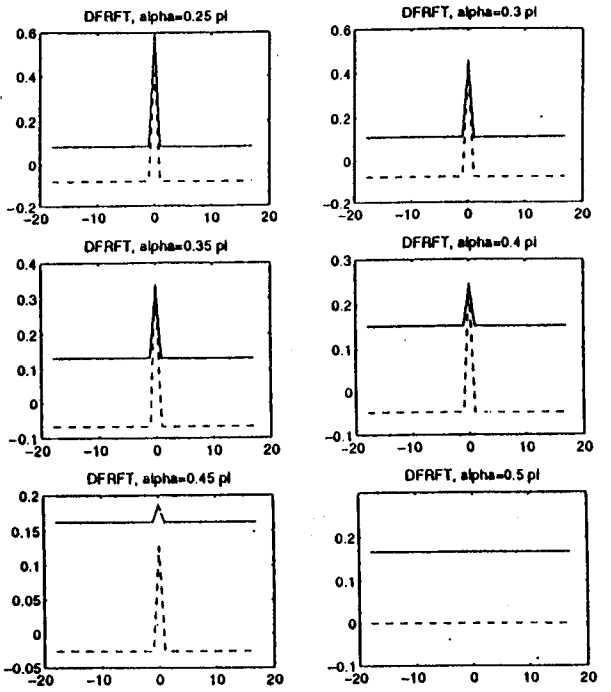

Figure 2 The conventional DFRFT of the unit sample function $\delta(n)$.

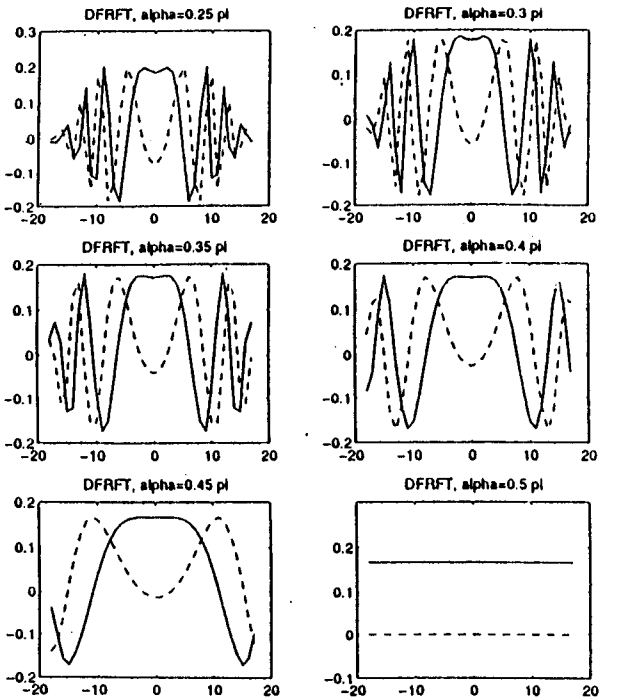

Figure 3 The new DFRFT of the unit sample function $\delta(n)$.