ONE-DIMENSIONAL MODE ALGORITHM FOR TWO-DIMENSIONAL FREQUENCY ESTIMATION

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Abstract
This paper describes how the computationally efficient one-dimensional MODE (1D-MODE) algorithm can be used to estimate the frequencies of two-dimensional complex sinusoids. We show that the 1D-MODE algorithm is computationally more efficient than the asymptotically statistically efficient 2D-MODE algorithm, especially when the numbers of spatial measurements are large. We find that the 1D-MODE algorithm is asymptotically statistically efficient for high signal-to-noise ratio. We also show that although 1D-MODE is no longer statistically efficient when the number of temporal snapshots is large, the performance of 1D-MODE can still be very close to that of the 2D-MODE under mild conditions. Numerical examples comparing the performances of the 1D-MODE and 2D-MODE algorithms are also presented.

1. INTRODUCTION

In [1], we presented a two-dimensional MODE (2D-MODE) algorithm for estimating 2-D frequencies. There are many applications for 2-D frequency estimation, which include angle-of-arrival estimation with a 2-D sensor array and synthetic aperture radar imaging [1]. Compared with the exact maximum likelihood estimator, the 2D-MODE algorithm avoids the multidimensional search over the parameter space [2]. Yet 2D-MODE has been shown to be statistically efficient under either the assumption that the number of temporal snapshots is large or the signal-to-noise ratio (SNR) is high.

The purpose of this paper is to describe how the computationally efficient one-dimensional MODE (1D-MODE) algorithm [3] can be used to estimate the frequencies of 2-D complex sinusoids. We show that the 1D-MODE algorithm is computationally more efficient than the 2D-MODE, especially when the numbers of spatial measurements are large. We also find the 1D-MODE algorithm is statistically efficient for high signal-to-noise ratio (SNR). Even thought the 1D-MODE algorithm is no longer statistically efficient when the number of temporal snapshots is large, its performance can still be very close to that of the 2D-MODE under mild conditions. Numerical examples comparing the performances of the 1D-MODE and 2D-MODE algorithms are included in this paper.

2. PROBLEM FORMULATION

Consider the following model of 2-D complex sinusoids in additive noise:

$$y_{m,\overline{m}}(t_n) = \sum_{k=1}^{K}\sum_{\overline{k}=1}^{\overline{K}} \alpha_{k,\overline{k}}(t_n)e^{j(\omega_k m+\mu_{\overline{k}}\overline{m})} + e_{m,\overline{m}}(t_n), \quad (1)$$

where $m = 1,2,\cdots,M$, $\overline{m} = 1,2,\cdots,\overline{M}$, and $n = 1,2,\cdots,N$. We refer to M ($M > K$) and $\overline{M}$ ($\overline{M} > \overline{K}$) as the numbers of *spatial measurements*, and to N as the number of *temporal snapshots*. The additive noise $e_{m,\overline{m}}(t_n)$ is assumed to be a complex Gaussian random process with zero-mean and

$$E\{e_{m,\overline{m}}(t_{n_1})e^*_{m,\overline{m}}(t_{n_2})\} = \sigma^2\delta_{n_1,n_2}, \quad (2)$$

where $(\cdot)^*$ denotes the complex conjugate and δ_{n_1,n_2} denotes the Kronecker delta. The $e_{m,\overline{m}}(t_n)$, $m = 1,2,\cdots,M$, $\overline{m} = 1,2,\cdots,\overline{M}$, are also assumed to be independent of each other and the complex sinusoids. The complex amplitudes $\alpha_{k,\overline{k}}(t_n)$, $k = 1,2,\cdots,K$, $\overline{k} = 1,2,\cdots,\overline{K}$, may be modeled either as the *stochastic* (or unconditional) signal model or as the *deterministic* (or conditional) signal model [4, 5].

Let $\mathbf{Y}(t_n)$ and $\mathbf{E}(t_n)$ be $M\times\overline{M}$ matrices whose $(m,\overline{m})$th elements, respectively, are $y_{m,\overline{m}}(t_n)$ and $e_{m,\overline{m}}(t_n)$. Define $\mathbf{X}(t_n)$ to be a $K\times\overline{K}$ matrix whose $(k,\overline{k})$th element is $\alpha_{k,\overline{k}}(t_n)$. Let

$$\mathbf{A} = [\ \mathbf{a}(\omega_1)\ \ \cdots\ \ \mathbf{a}(\omega_K)\ ], \quad (3)$$

$$\mathbf{a}(\omega_k) = [\ e^{j\omega_k}\ \ \cdots\ \ e^{jM\omega_k}\ ]^T, \quad (4)$$

$$\mathbf{B} = [\ \mathbf{b}(\mu_1)\ \ \cdots\ \ \mathbf{b}(\mu_{\overline{K}})\ ], \quad (5)$$

and

$$\mathbf{b}(\mu_{\overline{k}}) = [\ e^{j\mu_{\overline{k}}}\ \ \cdots\ \ e^{j\overline{M}\mu_{\overline{k}}}\ ]^T, \quad (6)$$

where $k = 1,2,\cdots,K; \overline{k} = 1,2,\cdots,\overline{K}$; and $(\cdot)^T$ denotes the transpose. Then $\mathbf{Y}(t_n)$ can be written as

$$\mathbf{Y}(t_n) = \mathbf{A}\mathbf{X}(t_n)\mathbf{B}^T + \mathbf{E}(t_n). \quad (7)$$

The problem of interest herein is to estimate ω_1, ω_2, $\cdots$, ω_K and μ_1, μ_2, $\cdots$, $\mu_{\overline{K}}$ from $\mathbf{Y}(t_n)$, $n = 1,2,\cdots,N$.

3. 2-D FREQUENCIES ESTIMATES WITH 1D-MODE

First consider using 1D-MODE to estimate $\omega = [\omega_1,\omega_2,\cdots,\omega_K]^T$. Let

$$\hat{\mathbf{R}}_\omega = \frac{1}{N}\sum_{n=1}^{N}\mathbf{Y}(t_n)\mathbf{Y}^H(t_n), \quad (8)$$

where $(\cdot)^H$ denotes the complex conjugate transpose and $\hat{\mathbf{R}}_\omega$ is the estimate of the following spatial covariance matrix:

$$\mathbf{R}_\omega = E\left\{\mathbf{Y}(t_n)\mathbf{Y}^H(t_n)\right\} = \mathbf{A}\mathbf{P}_\omega\mathbf{A}^H + \overline{M}\sigma^2\mathbf{I}, \quad (9)$$

where $E\{\cdot\}$ denotes the expectation and

$$\mathbf{P}_\omega = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} E\left\{\mathbf{X}(t_n)\mathbf{B}^T\mathbf{B}^*\mathbf{X}^H(t_n)\right\}, \qquad (10)$$

with $(\cdot)^*$ denoting the complex conjugate. Note that we use (10) to accommodate both the deterministic and stochastic signal models. The 1D-MODE algorithm [3], or, in a related form, WSF [6], can be applied to $\hat{\mathbf{R}}_\omega$ to obtain the estimate of ω, as shown below.

Let the columns in $\hat{\mathbf{E}}_\omega^s$ be the signal subspace eigenvectors of $\hat{\mathbf{R}}_\omega$ that correspond to the $\tilde{K}_\omega$ largest eigenvalues of $\hat{\mathbf{R}}_\omega$, where

$$\tilde{K}_\omega = \min[\overline{M}N, \mathrm{rank}(\mathbf{P}_\omega)]. \qquad (11)$$

We assume that $\tilde{K}_\omega$ is known. (If $\tilde{K}_\omega$ is unknown, it can be estimated from the data as described, for example, in [7].) Further, let $\hat{\mathbf{\Lambda}}_\omega$ be a diagonal matrix with diagonal elements $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{\tilde{K}_\omega}$, which are the $\tilde{K}_\omega$ largest eigenvalues of $\hat{\mathbf{R}}_\omega$, and

$$\hat{\mathbf{\Lambda}}_\omega^s = \hat{\mathbf{\Lambda}}_\omega - \overline{M}\hat{\sigma}^2\mathbf{I}, \qquad (12)$$

with $\mathbf{I}$ denoting the identity matrix and

$$\begin{aligned}\hat{\sigma}^2 &= \tfrac{1}{(M-\tilde{K}_\omega)\overline{M}} \textstyle\sum_{i=\tilde{K}_\omega+1}^{M} \hat{\lambda}_i \\ &= \tfrac{1}{(M-\tilde{K}_\omega)\overline{M}} \left[\mathrm{tr}\left(\hat{\mathbf{R}}_\omega\right) - \textstyle\sum_{i=1}^{\tilde{K}_\omega} \hat{\lambda}_i\right].\end{aligned} \qquad (13)$$

It is worth noting that the involved computational burden to evaluate $\hat{\mathbf{E}}_\omega^s$, $\hat{\mathbf{\Lambda}}_\omega$, and $\hat{\mathbf{\Lambda}}_\omega^s$ is of the order $O(M^2)$, since usually $\tilde{K}_\omega \ll M$, and hence much reduced compared with what would be required for a full eigendecomposition.

The 1D-MODE (or WSF) estimate $\hat{\omega}$ of ω can be obtained by minimizing the following function:

$$f(\omega) = \mathrm{tr}\left[\mathbf{P}_{\mathbf{A}}^{\perp}(\omega)\hat{\mathbf{E}}_\omega^s \left(\hat{\mathbf{\Lambda}}_\omega^s\right)^2 \hat{\mathbf{\Lambda}}_\omega^{-1} \left(\hat{\mathbf{E}}_\omega^s\right)^H\right], \qquad (14)$$

where, for some matrix $\mathbf{Z}$, the symbol $\mathbf{P}_{\mathbf{Z}}^{\perp}$ stands for the orthogonal projector onto the null space of $\mathbf{Z}^H$. To compute the estimate of ω without searching over the parameter space, the projector $\mathbf{P}_{\mathbf{A}}^{\perp}$ above must be reparameterized in terms of the coefficients of the so-called "linear predictor" polynomial [1, 3, 5]. We remark that $\hat{\omega}$ is a consistent estimate of ω for either large N or high SNR [5, 8].

Let

$$\hat{\mathbf{R}}_\mu = \frac{1}{N}\sum_{n=1}^{N} \mathbf{Y}^T(t_n)\mathbf{Y}^*(t_n) \qquad (15)$$

be the estimate of the following spatial covariance matrix:

$$\mathbf{R}_\mu = E\left\{\mathbf{Y}^T(t_n)\mathbf{Y}^*(t_n)\right\} = \mathbf{B}\mathbf{P}_\mu\mathbf{B}^H + M\sigma^2\mathbf{I}, \qquad (16)$$

where

$$\mathbf{P}_\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} E\left\{\mathbf{X}^T(t_n)\mathbf{A}^T\mathbf{A}^*\mathbf{X}^*(t_n)\right\}, \qquad (17)$$

Similarly, the 1D-MODE algorithm can be applied to $\hat{\mathbf{R}}_\mu$ to obtain the estimate $\hat{\mu}$ of μ.

We remark that the amount of computations required by the 2D-MODE algorithm is $O(M^2\overline{M}^2N)$ and that required by 1D-MODE is $O(M\overline{M}(M+\overline{M})N)$. Since the 2D-MODE algorithm requires the computation and eigendecomposition of an $M\overline{M} \times M\overline{M}$ matrix $\hat{\mathbf{R}}$ [1], where

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N} \mathrm{vec}\left[\mathbf{Y}^T(t_n)\right] \mathrm{vec}^H\left[\mathbf{Y}^T(t_n)\right], \qquad (18)$$

with $\mathrm{vec}(\cdot)$ denoting stacking all columns of a matrix into a single column vector, while both $\hat{\mathbf{R}}_\omega$ and $\hat{\mathbf{R}}_\mu$ can be formed from only the diagonal blocks of $\hat{\mathbf{R}}$. Thus for large M and $\overline{M}$, 1D-MODE requires much less computations than 2D-MODE.

4. STATISTICAL PERFORMANCE ANALYSIS

4.1. The Case of High SNR

In the case of high SNR, i.e., $\sigma \ll 1$ (whereas $\mathbf{P}_\omega$ and $\mathbf{P}_\mu$ are $O(1)$), the asymptotic covariance matrices of the estimate $\hat{\omega}$ of ω and $\hat{\mu}$ of μ are respectively given by

$$E\left\{(\hat{\omega}-\omega)(\hat{\omega}-\omega)^T\right\} = \frac{\sigma^2}{2N}\left[\mathrm{Re}\left(\mathbf{H}_\omega \odot \hat{\mathbf{P}}_\omega^T\right)\right]^{-1}, \quad (19)$$

$$E\left\{(\hat{\mu}-\mu)(\hat{\mu}-\mu)^T\right\} = \frac{\sigma^2}{2N}\left[\mathrm{Re}\left(\mathbf{H}_\mu \odot \hat{\mathbf{P}}_\mu^T\right)\right]^{-1}, \quad (20)$$

where $\odot$ denotes the Hadamard-Schur matrix product (i.e., elementwise multiplication),

$$\hat{\mathbf{P}}_\omega = \frac{1}{N}\sum_{n=1}^{N} \mathbf{X}(t_n)\mathbf{B}^T\mathbf{B}^*\mathbf{X}^H(t_n), \qquad (21)$$

$$\hat{\mathbf{P}}_\mu = \frac{1}{N}\sum_{n=1}^{N} \mathbf{X}^T(t_n)\mathbf{A}^T\mathbf{A}^*\mathbf{X}^*(t_n), \qquad (22)$$

$$\mathbf{H}_\omega = \mathbf{D}_\omega^H\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D}_\omega, \qquad (23)$$

with the kth column of $\mathbf{D}_\omega$ being $\partial\mathbf{a}(\omega_k)/\partial\omega_k$, and

$$\mathbf{H}_\mu = \mathbf{D}_\mu^H\mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{D}_\mu, \qquad (24)$$

with the kth column of $\mathbf{D}_\mu$ being $\partial\mathbf{b}(\mu_k)/\partial\mu_k$.

It has also been shown in [1] that the asymptotic (for SNR $\gg 1$) covariance matrices of the estimate of ω and μ obtained with 2D-MODE are equal to the corresponding deterministic Cramer-Rao bound (CRB) given by

$$\left[\left(\mathbf{CRB}_\omega^d\right)^{-1}\right]_{ij} = (2N/\sigma^2)\mathrm{Re}\left\{\mathrm{tr}\left(\left[\left((\mathbf{A}_j')^H\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{A}_i'\right) \otimes (\mathbf{B}^H\mathbf{B})\right]\hat{\mathbf{S}}\right)\right\}, \qquad (25)$$

$$\left[\left(\mathbf{CRB}_\mu^d\right)^{-1}\right]_{ij} = (2N/\sigma^2)\mathrm{Re}\left(\mathrm{tr}\left\{\left[(\mathbf{A}^H\mathbf{A}) \otimes \left((\mathbf{B}_j')^H\mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{B}_i'\right)\right]\hat{\mathbf{S}}\right\}\right), \qquad (26)$$

where $\otimes$ denotes the Kronecker product, $\mathbf{A}_i' = \partial\mathbf{A}(\omega_i)/\partial\omega_i$, $\mathbf{B}_i' = \partial\mathbf{B}(\mu_i)/\partial\mu_i$, and $\hat{\mathbf{S}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{s}(t_n)\mathbf{s}^H(t_n)$ with $\mathbf{s}(t_n) = \mathrm{vec}[\mathbf{X}^T(t_n)]$.

From a straight forward computation of Equations (19) and (20), we have

$$E\left\{(\hat{\omega}-\omega)(\hat{\omega}-\omega)^T\right\} = \mathbf{CRB}_\omega^d, \qquad (27)$$

and

$$E\left\{(\hat{\mu}-\mu)(\hat{\mu}-\mu)^T\right\} = \mathbf{CRB}_\mu^d. \qquad (28)$$

Thus the estimates $\hat{\omega}$ and $\hat{\mu}$ obtained with the 1D-MODE asymptotically (for high SNR) achieve the CR-bounds in (25) and (26), respectively, which means that the 1D-MODE is an asymptotically (for SNR $\gg$ 1) statistically efficient estimator for estimating the 2-D frequencies.

4.2. The Case of Large N

Similar to the case of high SNR, the asymptotic covariance matrices of $\hat{\omega}$ and $\hat{\mu}$ in the case of large N are respectively given by

$$E\left\{(\hat{\omega}-\omega)(\hat{\omega}-\omega)^T\right\} = \frac{\sigma^2}{2N}\left[\mathrm{Re}\left(\mathbf{H}_\omega \odot \mathbf{V}_\omega^T\right)\right]^{-1}, \quad (29)$$

and

$$E\left\{(\hat{\mu}-\mu)(\hat{\mu}-\mu)^T\right\} = \frac{\sigma^2}{2N}\left[\mathrm{Re}\left(\mathbf{H}_\mu \odot \mathbf{V}_\mu^T\right)\right]^{-1}, \quad (30)$$

where

$$\mathbf{V}_\omega = \mathbf{P}_\omega \mathbf{A}^H \mathbf{R}_\omega^{-1} \mathbf{A} \mathbf{P}_\omega, \quad (31)$$

and

$$\mathbf{V}_\mu = \mathbf{P}_\mu \mathbf{B}^H \mathbf{R}_\mu^{-1} \mathbf{B} \mathbf{P}_\mu. \quad (32)$$

It has been shown in [1] that the large-sample covariance matrices of the estimates of ω and μ obtained with 2D-MODE are equal to the corresponding stochastic CRBs given by

$$\left[(\mathbf{CRB}_\omega^s)^{-1}\right]_{ij} = (2N/\sigma^2)\mathrm{Re}\left(\mathrm{tr}\left\{\left[\left((\mathbf{A}_j')^H \mathbf{P}_{\mathbf{A}}^\perp \mathbf{A}_i'\right) \otimes \left(\mathbf{B}^H\mathbf{B}\right)\right] \mathbf{S}\left(\mathbf{A}^H \otimes \mathbf{B}^H\right)\mathbf{R}^{-1}\left(\mathbf{A}\otimes\mathbf{B}\right)\mathbf{S}\right\}\right), \quad (33)$$

and

$$\left[(\mathbf{CRB}_\mu^s)^{-1}\right]_{ij} = (2N/\sigma^2)\mathrm{Re}\left(\mathrm{tr}\left\{\left[\left(\mathbf{A}^H\mathbf{A}\right) \otimes \left((\mathbf{B}_j')^H \mathbf{P}_{\mathbf{B}}^\perp \mathbf{B}_i'\right)\right] \mathbf{S}\left(\mathbf{A}^H \otimes \mathbf{B}^H\right)\mathbf{R}^{-1}\left(\mathbf{A}\otimes\mathbf{B}\right)\mathbf{S}\right\}\right), \quad (34)$$

respectively, where

$$\mathbf{S} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} E\left\{\mathbf{s}(t_n)\mathbf{s}^H(t_n)\right\}, \quad (35)$$

and

$$\mathbf{R} = E\left\{\mathrm{vec}\left[\mathbf{Y}^T(t_n)\right]\mathrm{vec}^H\left[\mathbf{Y}^T(t_n)\right]\right\}. \quad (36)$$

In this case, the 1D-MODE is no longer an asymptotically statistically efficient estimator for estimating the 2-D frequencies. According to the general theory of the CR-bounds, we have

$$E\left\{(\hat{\omega}-\omega)(\hat{\omega}-\omega)^T\right\} \geq \mathbf{CRB}_\omega^s, \quad (37)$$

and

$$E\left\{(\hat{\mu}-\mu)(\hat{\mu}-\mu)^T\right\} \geq \mathbf{CRB}_\mu^s. \quad (38)$$

The numerical examples given in the following section show that the larger the M ($\bar{M}$) or the higher the SNR, the smaller the difference between $E\left\{(\hat{\omega}-\omega)(\hat{\omega}-\omega)^T\right\}$ ($E\left\{(\hat{\mu}-\mu)(\hat{\mu}-\mu)^T\right\}$) and $\mathbf{CRB}_\omega^s$ ($\mathbf{CRB}_\mu^s$).

4.3. Further Comments

We remark that when $\mathbf{X}$ in (7) is a diagonal matrix, $y_{m,\bar{m}}(t_n)$ can be modeled with the following data model [1]:

$$y_{m,\bar{m}}(t_n) = \sum_{\breve{k}=1}^{\breve{K}} \alpha_{\breve{k}}(t_n) e^{j(\omega_{\breve{k}} m + \mu_{\breve{k}} \bar{m})} + e_{m,\bar{m}}(t_n). \quad (39)$$

For this case, the 1D-MODE approach can again be used to estimate the 2-D frequencies. Yet it can be shown that serious performance degradation can occur when any of the 1-D approaches (including 1D-MODE) is used with (39), which makes our results even more unexpected. An intuitive explanation is that when $\mathbf{X}$ is a diagonal matrix, 1-D processing does not exploit all of the information available and hence lacks the statistical efficiency. When $\mathbf{X}$ is a full matrix, which is the case we assume, there is no structural information that is missed by 1-D processing and hence there is no performance degradation under mild conditions.

We also remark that the 1D-MODE approach can be applied to data with non-Gaussian noise without any modification. Its asymptotic covariance matrices will be the same. The CRB matrix for the non-Gaussian case will be different from the one for the Gaussian case, but the CRB matrix computed under the Gaussian assumption remains the lower bound for a *large* class of estimators whose asymptotic covariance matrices do not depend on the data distribution (for instance all estimators based on second-order statistics).

5. NUMERICAL RESULTS

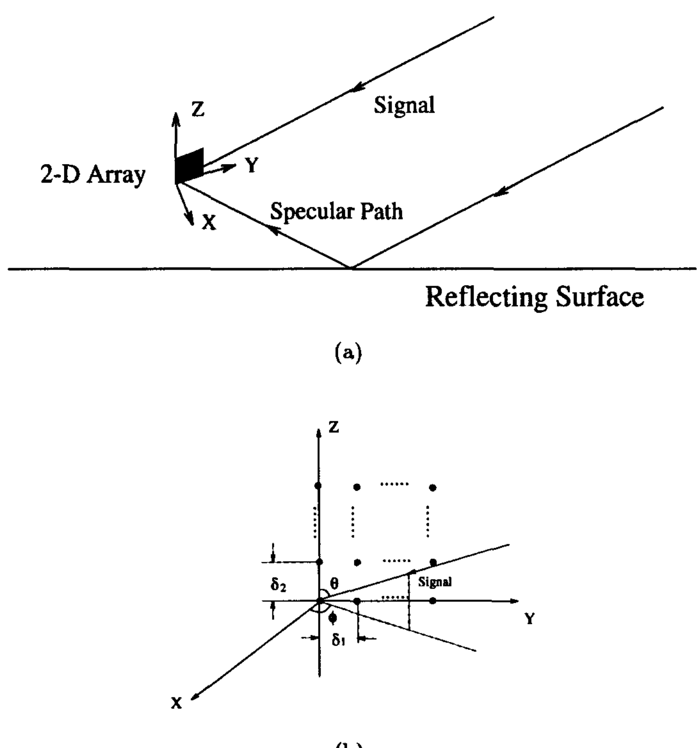
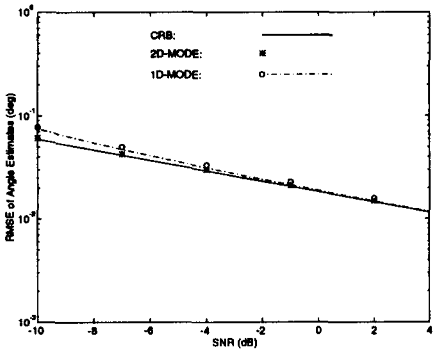
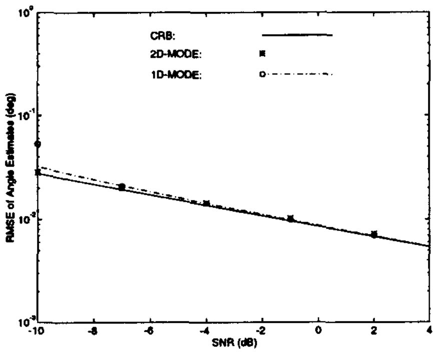


Figure 1. *Direction-of-arrival estimation with a 2-D array.*

We present below an example showing the performance of the 1D-MODE algorithm. The empirical results of the algo-



(a)



(b)

Figure 2. *Root-mean-squared errors of angle estimates as a function of SNR when the direct and reflected signals arrive from* $(45°, 85°)$ *and* $(45°, 95°)$*, respectively. The correlation coefficient between the direct and reflected signals is* 0.99*. Further,* $M = 10$*,* $\overline{M} = 16$*, and* $N = 500$*. (a) The estimates of the azimuth angle* ϕ*. (b) The estimates of the elevation angle* θ*.*

rithms are obtained from 50 independent Monte-Carlo trials and are compared with the results of the 2D-MODE algorithm and with the corresponding theoretical asymptotic statistical performances.

We consider the case of large N, which occurs in two-dimensional angle estimation by means of an $M \times \overline{M}$ rectangular uniform linear array when the incident angles are related to each other [1]. The incident signals are assumed to be narrowband plane waves. In this application, N denotes the number of snapshots taken at the output of the array. Consider the case shown in Figure 1 where a vertical 2-D rectangular uniform linear array with $M = 10$ and $\overline{M} = 16$ is used to estimate the 2-D incident angles of a signal arriving from a low angle relative to a smooth reflecting surface such as the calm sea. Assume that the original signal arrives from (ϕ_1, θ_1), where ϕ_1 and θ_1 denote the azimuth and elevation angles of the signal, as shown in Figure 1. Then its reflected signal arrives from $(\phi_2, \theta_2) = (\phi_1, 180° - \theta_1)$. The 2-D incident angles can be calculated from the $\{\omega_k\}$ and $\{\mu_{\overline{k}}\}$ in Equation (1) by:

$$\omega_k = \frac{2\pi\delta_1}{\lambda_0} \sin\theta_k \sin\phi_k, \quad k = 1, \tag{40}$$

and

$$\mu_{\overline{k}} = \frac{2\pi\delta_2}{\lambda_0} \cos\theta_{\overline{k}}, \quad \overline{k} = 1, 2, \tag{41}$$

where λ_0 denotes the wavelength of the incident signals. In our examples, the SNR of the reflected signal is assumed to be 3dB less than that of the direct signal. Further, the correlation coefficient between the direct and the reflected signals is 0.99. The spacings δ_1 and δ_2 between two adjacent sensors in the array are assumed equal to a half wavelength. The asymptotic variances of the estimates of ϕ and θ are readily obtained from the asymptotic variances of the estimates of ω_k and $\mu_{\overline{k}}$ given in Section 4 and the Equations (40) and (41) relating ϕ_k and θ_k to ω_k and $\mu_{\overline{k}}$.

Figure 2 shows the root-mean-squared errors (RMSEs) of the angle estimates and the corresponding asymptotic (for large N) statistical performances of the 1D-MODE and 2D-MODE algorithms as a function of the SNR of the direct signal when $\phi = 45°$, $\theta = 85°$, and $N = 500$. As expected, as the SNR increases, the performance of 1D-MODE approaches that of the 2D-MODE, whose asymptotic statistical performance is also equal to the corresponding CR-bound. The performance of 2D-MODE is slightly better than that of 1D-MODE only when the SNR is very small. Yet the amount of computations needed by 1D-MODE was only about 7% of that required by 2D-MODE in our simulations. Further, as M and $\overline{M}$ increase, more computational savings can be achieved by using 1D-MODE.

REFERENCES

[1] J. Li, P. Stoica, and D. Zheng, "An efficient algorithm for two-dimensional frequency estimation," *Multidimensional Systems and Signal Processing*, vol. 7, pp. 151–178, April 1996.

[2] C. R. Rao, L. C. Zhao, and B. Zhou, "Maximum likelihood estimation of 2-d superimposed exponential signals," *IEEE Transactions on Signal Processing*, vol. 42, pp. 1795–1802, July 1994.

[3] P. Stoica and K. C. Sharman, "Maximum likelihood methods for direction-of-arrival estimation," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-38, pp. 1132–1143, July 1990.

[4] P. Stoica and A. Nehorai, "Performance study of conditional and unconditional direction-of-arrival estimation," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-38, pp. 1783–1795, October 1990.

[5] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "Exact and Large Sample ML Techniques for Parameter Estimation and Detection in Array Processing." In *Radar Array Processing*, S. Haykin *et. al.* (eds.), Chapter 4. New York, NY: Springer-Verlag Inc., 1993.

[6] M. Viberg and B. Ottersten, "Sensor array processing based on subspace fitting," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-39, pp. 1110–1121, May 1991.

[7] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-33, pp. 387–392, April 1985.

[8] M. Viberg, "Sensitivity of parametric direction finding to colored noise fields and undermodeling," *Signal Processing*, vol. 34, pp. 207–222, 1993.