

INVERSE EIGENVALUE PROBLEM FOR SINUSOIDAL FREQUENCY ESTIMATION

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ABSTRACT

The estimation of the frequencies of sinusoids in noise is a very common problem. This paper addresses the estimation of sinusoids in a low SNR environment. This sinusoidal frequency estimation problem can be used to find the carrier frequencies and baud rates of communication waveforms after some appropriate nonlinearity.

If the underlying signal model is sinusoids in white Gaussian noise and we use the forward/backward prediction framework, then the forward/backward prediction equations force a Toeplitz/Hankel structure on the data matrix. If there are M distinct sinusoids in the data and no noise, then the data matrix has rank M .

Cadzow and Wilkes [1] enhance a noisy data matrix by enforcing both the structure and the rank of the data matrix, before solving for the coefficient vector of the prediction problem. Besides the Toeplitz/Hankel structure, I also enforce the estimated singular values of the data matrix. Using more information extracted from the original data matrix extends the threshold to lower SNR values.

1. INTRODUCTION

In the following I propose a new algorithm for signal enhancement and parameter estimation. As an example the behavior of the algorithm to estimate the frequencies of multiple sinusoids in white Gaussian noise is given.

I introduce the problem giving a short overview of the structure and quantities involved in this particular signal modeling. The next section presents some results concerning the singular vectors of structured matrices based on random samples. The original algorithm and its improvement is in the following section. A numerical example is covered in depth in the last section.

2. STRUCTURES AND QUANTITIES

Given N samples of the waveform, the problem is setup as a forward/backward prediction problem on the data matrix. This is a standard technique for this problem [5, pg.367]. A sampled mix of sinusoids will satisfy the homogeneous equation:

$$a_0 x(n) + a_1 x(n-1) + \dots + a_p x(n-p) = 0, \text{ for } p+1 \leq n \leq N.$$

Arranging these equations in matrix form, one exhibits the Toeplitz structure.

$$\begin{pmatrix} x(p+1) & x(p) & \dots & x(1) \\ x(p+2) & x(p+1) & \dots & x(2) \\ \vdots & \vdots & & \vdots \\ x(N) & x(N-1) & \dots & x(N-p) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ p \end{pmatrix} = 0.$$

For sinusoids one can also write down the relationship for the backward prediction (the over bar is the complex conjugate):

$$\begin{pmatrix} \bar{x}(1) & \bar{x}(2) & \dots & \bar{x}(p+1) \\ \bar{x}(2) & \bar{x}(3) & \dots & \bar{x}(p+2) \\ \vdots & \vdots & & \vdots \\ \bar{x}(N-p) & \bar{x}(N-p+1) & \dots & \bar{x}(N) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ p \end{pmatrix} = 0.$$

Measuring parameters in noisy data, one uses as many equations as possible. The composite of the two matrices above is of Toeplitz and Hankel structure. This will be the data matrix $X = \begin{pmatrix} X_T \\ X_H \end{pmatrix}$. In the case of a noise free data matrix built on a mix of sinusoids, the rank of the matrix is equal to the number of different complex sinusoids in the signal. If one wants to estimate the parameters of M complex sinusoids given N noisy sample and $M \ll N$, one has to pick the dimension p for the data matrix. This dimension is often selected between $\frac{N}{3}$ and $\frac{N}{2}$.

3. SINGULAR VALUES OF STRUCTURED RANDOM MATRICES

The singular values of a matrix are the square root of the eigenvalues of the covariance matrix $R = X^* X$ (X^* is the transposed, complex conjugate of matrix X).

The eigenvalues λ_k of the covariance matrix are a continuous function of the random variable x_m . The derivative of the eigenvalues λ_k with respect to the random variables x_m is from [2]

$$\frac{\partial \lambda_k}{\partial x_m} = u^* \frac{\partial R}{\partial x_m} u.$$

Being the square roots of the eigenvalues the singular values are also continuous functions of the random variables x_m that represent our noisy signal measurement. If there are multiple eigenvalues, the eigenvalues can not be guaranteed differentiable.

The mean and the standard deviation of the singular

values of a structured matrix based on random variables x_k are well defined. With no signal present, the mean and the standard deviation of the singular values of the data matrix X was numerically computed. For an 80×21 data matrix X I show the mean of the singular values as a solid line in Figure 1. The error bars reflect one standard deviation around the mean.

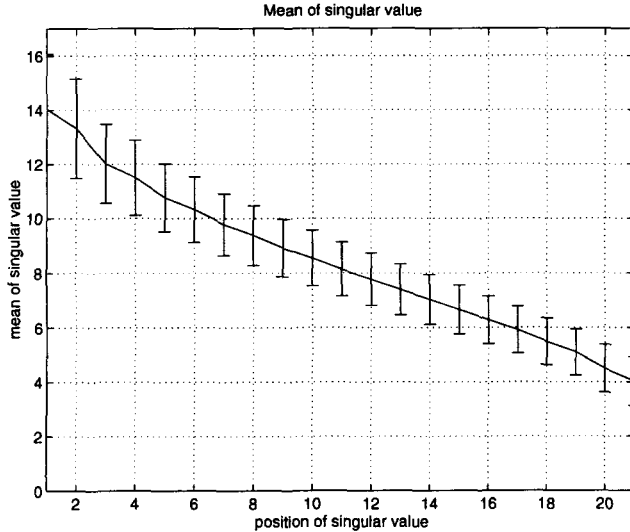


Figure 1. Mean and Std. Deviation of the Singular Values

A data matrix built from random samples exhibits a well defined mean of its sorted singular values. Thus one can subtract the noise contribution from the singular values recovered from the data matrix to get an estimate of the signal singular values.

4. ALGORITHMIC DEVELOPMENT

4.1. Standard Enhancement

Cadzow and Wilkes [1] enhance the signal using an iterative technique. This technique is very close to the Inverse Eigenvalue Problem (IEP) [2]. The IEP is a standard problem in mathematics.

This is the signal enhancement algorithm of Cadzow and Wilkes to find M distinct sinusoids:

Given the data matrix $X = \begin{pmatrix} X_T \\ X_H \end{pmatrix}$.

a) Compute the SVD of X and find the M th rank approximation by keeping the largest M singular values and setting the remaining singular values to zero.

b) Compute the Toeplitz/Hankel approximation to the low-rank matrix by averaging over the respective diagonals. Since step b) will increase the rank of the matrix, iterate step a) and b) till some convergence is achieved.

Once convergence is achieved towards a low-rank, structured matrix, one can use the SVD method [3]. The rank M enhanced data matrix \hat{X} is decomposed into its first column \hat{x}_1 and the remaining p columns \hat{X}_r : $\hat{X} = \begin{pmatrix} \hat{x}_1 & \hat{X}_r \end{pmatrix}$.

The coefficient vector $a = \begin{pmatrix} 1 \\ a_r \end{pmatrix}$ is computed from:

$$a_r = -\hat{X}^\dagger \hat{x}_1,$$

where \dagger denotes the pseudo inverse. The frequencies are detected using the function $d(f)$:

$$d(f) = \frac{1}{\left| \sum_{n=0}^p a_n e^{j2\pi f n} \right|}.$$

4.2. Removing the Noise Floor

The new algorithm improves on the above in the following way:

Instead of letting the algorithm pick the singular values of the low-rank approximation of the Toeplitz/Hankel matrix at each step, the M largest singular values are computed at the first step and then stay fixed for the rest of the iteration. This restricts the movement of the singular values and utilizes the given information more fully than the above algorithm.

A straight forward implementation of this algorithm will actually result in a slight loss of performance compared to the Cadzow/Wilkes algorithm. The important factor to note is that in a low SNR environment the largest M singular values are also corrupted by noise.

One heuristic way to estimate the influence of the noise on the largest singular values is to compute the SVD for noise realizations without the signal repeatedly. If one wants to compute the parameters of M sinusoids, then $p+1-M$ singular values are based on the noise only. The smaller $p+1-M$ singular values can be used to estimate the noise contribution to the M larger singular values.

Here I considered only a linear relationship between the singular values. Collecting the ordered singular values in a matrix S , where the rows of the matrix correspond to one noise realization and SVD evaluation. Then the matrix $S = \begin{pmatrix} S_L & S_S \end{pmatrix}$ can be decomposed into its first M columns corresponding to the M larger singular values in S_L and the smaller ones in S_S . To estimate the larger singular value on the basis of the smaller singular values one introduces the matrix B (I_M is the identity matrix of size M): $B = \begin{pmatrix} I_M \\ -S_S^\dagger * S_L \end{pmatrix}$. Given a row vector s of the singular values, the signal singular values are computed as: $\hat{s} = (sB \ 0_{1 \times p+1-M})$. The singular values have to be positive and ordered. If the above estimation violates this constraint, then I replace the estimated singular value that violates these constraints. Substitute the estimated singular value by the sum of the difference of the original noisy singular values and the last estimated singular value, that satisfied the constraints. This is done from the smallest estimated singular value to the largest, to ensure the ordering of the singular values.

This gives the following algorithm (Here $\text{diag}(s)$ is the diagonal matrix D built on the vector s):

Compute the SVD of the data matrix $X = U_0 \text{diag}(s_0) V_0^*$ and estimate the signal singular values $\hat{s}_{\text{Signal}} = (s_0 B \ 0_{1 \times p+1-M})$.

For $k=1$ to L do {

a) Construct an enhanced data matrix \hat{X} using the estimated signal singular values and the corresponding singular vectors from the last SVD computation.

$$X_k = U_{k-1} * \text{diag}(\hat{s}_{\text{Signal}}) V_{k-1}^*.$$

- b) Compute the Toeplitz/Hankel approximation to the low-rank matrix by averaging over the respective diagonals.
- c) Compute the SVD of the matrix $X_k = U_k \text{diag}(s_k) V_k^*$.

Construct an enhanced data matrix \hat{X} using the estimated signal singular values and the corresponding singular vectors from the last SVD computation.

$$X_{L+1} = U_L \text{diag}(\hat{s}_{\text{Signal}}) V_L^*.$$

As in the standard enhancement, the enhanced data matrix \hat{X}_{L+1} is used to find the coefficient vector a and the frequencies f by evaluating the detection function $d(f)$.

4.3. Inverse Eigenvalue Problem

In [2] Friedland et. al. give methods for the solution of inverse eigenvalue methods. This application differs from [2] in that instead of the eigenvalues, the singular values of the data matrix X are given. These are the eigenvalues of the autocorrelation matrix $R = X^H X$. (X^H is the conjugate complex transpose of the matrix X .)

Similar to the IEP, the structure of the data matrix imposes constraints on the solution. The number of structural imposed constraints (=equations) vs. the number of available variables of a rank M approximation indicate an over determined system of equations. This over determination is the reason for the signal enhancement that is possible using these algorithms.

5. NUMERICAL EXAMPLE

5.1. Data

Similar to the example in [1], the algorithms were tested by the data:

$$x(n) = e^{j2\pi(0.2)n} + e^{j2\pi(0.21)n} + w(n) \quad n = 1, 2, \dots, 60,$$

where $w(n)$ is white noise. The frequencies of two sinusoids were estimated at each try. The SNR was varied in 1dB steps and the statistics were collected over 1000 noise realizations.

5.2. Algorithm

The algorithms tested are:

- 1) forward/backward linear prediction without signal enhancement
 - 2) standard enhancement as defined in [2], by enforcing a rank of 2 and a Toeplitz/Hankel structure on the data matrix,
 - 3) new algorithm, where the noise floor is removed from the first two singular values and the rest of the singular values is set to zero. These singular values as well as the Toeplitz/Hankel structure of the data matrix is enforced.
- The dimension of the combined Toeplitz/Hankel matrix was 80×21 . The singular values of a noise only data matrix were computed over 10,000 noise realizations. The two largest singular values are predicted from the next two singular values. The numerical values for the estimation of the noise component in the two largest singular values Σ_1^{Noise} and Σ_2^{Noise} were:

$$\begin{pmatrix} \Sigma_1^{\text{Noise}} \\ \Sigma_2^{\text{Noise}} \end{pmatrix} = \begin{pmatrix} 1.0564 & 0.1151 \\ 0.7949 & 0.3267 \end{pmatrix} \begin{pmatrix} \Sigma_3 \\ \Sigma_4 \end{pmatrix}.$$

If the subtraction of the noise component destroyed part or all of the ordering

$$\Sigma_1^{\text{NoiseRemoved}} > \Sigma_2^{\text{NoiseRemoved}} > 0,$$

then the estimates which did not conform to the ordering were replaced by differences of the original singular values.

5.3. Algorithm Evaluation

To evaluate the algorithms one wants to separate the influence of the:

- 1) Failure to resolve the frequencies due to low SNR and
- 2) Variance of the frequency estimate due to the presence of noise.

To differentiate between the two failure modes over a wide range of SNR, resolution of the sinusoid was defined as the two frequencies being resolved within an interval of $(0.2 - 10\sigma(\text{SNR}), 0.21 + 10\sigma(\text{SNR}))$. Here σ is the standard deviation of the frequencies predicted for the SNR by the Cramer Rao bound. If the resolution interval is selected too narrow, then:

- 1) the resolution of the frequencies will reflect the desired ability to resolve the frequencies correctly, but
- 2) the tails of the frequency distribution are cut off, resulting in a low estimate of the mean square error of the frequencies.

Thus the resolution interval is a compromise between the ability to correctly characterize resolution and estimate a well behaved standard deviation of the frequency distribution. The resolution interval vs SNR is graphed in Figure 2.

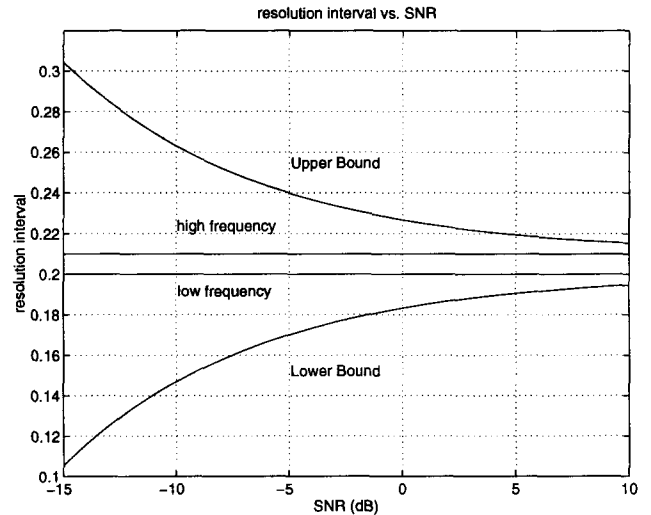


Figure 2. Resolution Interval over SNR

Cadzow and Wilkes [1] used a fixed interval of (0.17,0.24). This corresponds roughly to the interval given here at -5dB. Figure 3 gives the percentage of resolution for the different algorithms (higher is better). In a low SNR environment the new algorithm with the noise floor removed has about a 10dB advantage over the standard enhancement to reach corresponding resolution levels.

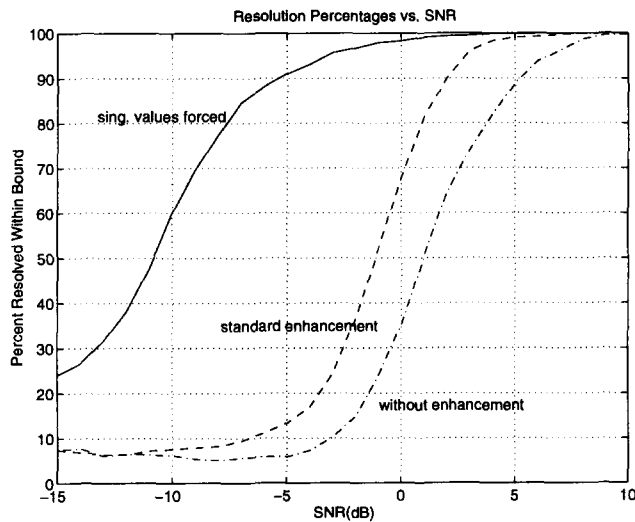


Figure 3. Resolution percentages

Removing the noise floor in the singular values allows also to track the Cramer-Rao bound better than the standard enhancement (Figure 4). As published in [1] the standard enhancement improves on the forward/backward prediction method.

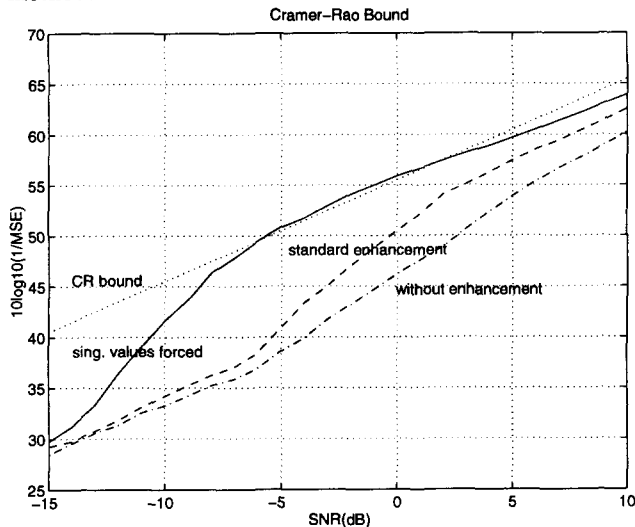


Figure 4. Cramer-Rao bound results

Due to the relatively small numbers of experiments, removing of the tails of the frequency distribution when considering only correctly resolved frequencies and approximations in computing the Cramer-Rao bound the results violate the Cramer-Rao bound in some instances [4, 6].

5.4. Time Sequence Sinusoidal Signal Enhancement

Enforcing the noise free estimates of the two largest singular values can be used to clean up noisy signals. An example is shown in Figure 5. The real part of the complex signal is drawn as a solid line, the imaginary part is drawn as a dashed line. The top sub plot shows the mix of sinusoids of amplitude 1 and the frequencies of 0.2 and 0.21 and the phases -30 and 30 degrees respectively. The middle sub

plot displays these sinusoids in AWGN for an overall SNR of -5dB. After 5 iterations of the algorithm, the signal is recovered by averaging over the entries in the data matrix corresponding to the Toeplitz/Hankel structure. The enhanced signal is shown in the bottom sub plot of Figure 5. The slight beat frequency in the signal is one indication that both frequency are resolved correctly in the resulting signal.

Sinusoids with Frequency 0.2 and 0.21 in -5dB AWGN

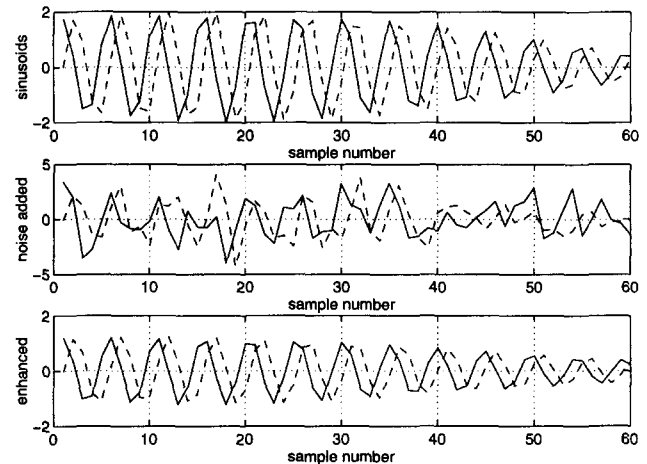


Figure 5. Time Sequence Sinusoidal Signal Enhancement

6. CONCLUSION

An improved algorithm for finding the frequencies of sinusoids in a low noise environment was presented. It relies on the matching computed singular values to a matrix with Toeplitz/Hankel structure. Its performance improves on the previously given algorithms.

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