ON SUBSPACE-BASED METHODS FOR FREQUENCY ESTIMATION OF RANDOM AMPLITUDE SINUSOIDAL SIGNALS

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ABSTRACT

Sinusoidal signals with random time-varying amplitude show up in many signal processing applications. Amplitude modulation results in degeneracy of the signal subspace, *i.e.* the signal subspace corresponding to one amplitude modulated sinusoid is no longer spanned by one vector. In this paper, we propose modifications of two subspace-based techniques, namely ESPRIT and MODE for estimating the center frequency of a sinusoidal signal with random time-varying $ARMA$ amplitude. Numerical simulations illustrate the good performance of the methods. Finally, a robustified scheme of the proposed methods is described and succesfully applied to real radar data.

1. PROBLEM STATEMENT

Analysis of sinusoidal signals with random or time varying amplitude appears to be of importance in many fields of signal processing. In the radar application, as soon as the target scintillates or with targets that cannot be considered as points, a random amplitude model is more appropriate than a constant amplitude one[1]. This problem is also encountered in the array processing context when the source signals impinge on the array over a spread of angles [2]. Most of the time, the amplitude may be viewed as a perturbation term (a "multiplicative noise") and a key problem consists of estimating the center frequency of such processes. From a spectral point of view, multiplicative noise is more difficult to cope with, random modulation causing a spread of the peak. Subspace-based techniques, which have proved to be powerful in the constant amplitude case, deteriorate in the case of a random amplitude. As a matter of fact, unless specific assumptions are made on the envelope, the signal subspace is degenerate, *i.e.* it has spread into a higher than one dimensional subspace. Consequently, subspace-based meth-ods, which rely on a clear discrimination between signal and noise subspaces are no longer suitable. Therefore, there is an interest in proposing subspace-based methods that properly handle the case of random amplitude signals. This is the aim of this paper. More precisely, we propose to use modifications of ESPRIT[3] and MODE[4] algorithms in order to estimate the frequency of an exponential signal with random $ARMA$ amplitude. Moreover, as we focus on the frequency estimation (and not on the $ARMA$ amplitude parameter estimation), we describe simplifications of these two methods, which reduce the associated computational burden.

2. MODEL AND FREQUENCY ESTIMATION

Let us consider the problem of estimating the frequency ω_0 from N samples of the following signal

$$x(t) = \alpha(t)e^{j(\omega_0 t + \varphi)} + w(t), \quad t = 1, ..., N \quad (1)$$

where φ is a deterministic phase, $w(t)$ is a sequence of zero-mean i.i.d. random variables with variance σ_w^2. In (1), $\alpha(t)$ is assumed to be a real-valued station-ary $ARMA(p,q)$ process, independent of $w(t)$. Let $S_\alpha(z) = \sigma^2 B(z)B(z^{-1})/\{A(z)A(z^{-1})\}$ denote the spec-trum of $\alpha(t)$ and define by $\{y_k = \rho_k e^{j\theta_k}\}_{k=1..p}$ the ze-roes of $A(z)$. It can be readily verified that the spec-trum $S_x(z)$ has the following $ARMA(p,p)$ form:

$$S_x(z) = \frac{\lambda^2 C(z)C(z^{-1})}{A(ze^{-j\omega_0})A(z^{-1}e^{j\omega_0})} \quad (2)$$

from which the following set of Yule-Walker equations can be written:

$$r_x(\tau) = -\sum_{k=1}^{p} a_k e^{jk\omega_0} r_x(\tau - k); \quad \tau > p \quad (3)$$

By inspection of (2), the poles of $S_x(z)$ are simply $z_k = y_k e^{j\omega_0} = \rho_k e^{j(\omega_0 + \theta_k)}$. This property forms the basis for the algorithms to be proposed. As a matter of fact, ω_0 can be extracted from the poles of the equivalent $ARMA$ spectral model as follows (herein $\arg(\rho e^{j\theta}) \overset{def}{=} \theta$):

$$\omega_0 = \frac{1}{p} \arg\left(\prod_{k=1}^{p} z_k\right) \qquad (4)$$

$$= \arg\left(\sum_{k=1}^{p} z_k\right) \qquad (5)$$

where we have used the fact that both $\prod_{k=1}^{p} y_k$ and $\sum_{k=1}^{p} y_k$ are real-valued. Hence, the problem of estimating ω_0 is reduced to that of $ARMA$ pole estimation. Let $\mathbf{R}$ denote the $M \times m$ covariance matrix with elements $\mathbf{R}(k,n) = r_x(m+k-n)$, with $m \geq p$ and $M \geq p$. By the Yule-Walker equations (3), $\mathbf{R}$ is of rank p. Let

$$\mathbf{R} = \mathbf{S}\mathbf{\Sigma}\mathbf{G}^H = \sum_{k=1}^{p} \sigma_k \mathbf{u}_k \mathbf{v}_k^H \qquad (6)$$

denote its SVD with $\mathbf{\Sigma} = diag(\sigma_1, ..., \sigma_p)$, $\mathbf{S} = [\mathbf{u}_1, ..., \mathbf{u}_p]$ and $\mathbf{G} = [\mathbf{v}_1, ..., \mathbf{v}_p]$.

2.1. ESPRIT

Let us partition $\mathbf{S}$ into all but last (first) row:

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_1 \\ - \end{pmatrix} = \begin{pmatrix} - \\ \mathbf{S}_2 \end{pmatrix} \qquad (7)$$

The basic property that underlies ESPRIT[3] can be formulated in eq. (8) (where $\mathbf{C}$ is a nonsingular transformation matrix):

$$\mathbf{S}_2 = \mathbf{S}_1 \mathbf{C}^{-1} \mathbf{D} \mathbf{C} \qquad (8)$$

where $\mathbf{D} = diag\left(\rho_1 e^{j(\theta_1 + \omega_0)}, ..., \rho_p e^{j(\theta_p + \omega_0)}\right)$. (8) is an overdetermined system of M linear equations that can be solved for $\mathbf{\Phi} = \mathbf{C}^{-1}\mathbf{D}\mathbf{C}$. Since $\mathbf{S}_1, \mathbf{S}_2$ have full rank, it follows that the solution is given by

$$\mathbf{\Phi} = \left(\mathbf{S}_1^H \mathbf{S}_1\right)^{-1} \left(\mathbf{S}_1^H \mathbf{S}_2\right) \qquad (9)$$

Moreover, $\mathbf{D}$ is the diagonal matrix of the eigenvalues of $\mathbf{\Phi}$. Therefore, the $ARMA$ poles can be estimated as the eigenvalues of $\left(\mathbf{S}_1^H \mathbf{S}_1\right)^{-1} \left(\mathbf{S}_1^H \mathbf{S}_2\right)$. Using (4), we have

$$\omega_0 = \frac{1}{p} \arg\left(\prod_{k=1}^{p} z_k\right) = \frac{1}{p} \arg\left(\det\left(\mathbf{D}\right)\right)$$

$$= \frac{1}{p} \arg\left(\det\left(\mathbf{\Phi}\right)\right) = \frac{1}{p} \arg\left(\det\left(\mathbf{S}_1^H \mathbf{S}_2\right)\right) \quad (10)$$

If we let $\widehat{\mathbf{S}}_1, \widehat{\mathbf{S}}_2$ denote sample estimates of $\mathbf{S}_1, \mathbf{S}_2$ respectively, then ω_0 is estimated as

$$\widehat{\omega}_0^d = \frac{1}{p} \arg\left(\det\left(\widehat{\mathbf{S}}_1^H \widehat{\mathbf{S}}_2\right)\right) \qquad (11)$$

which enables to avoid the second EVD required in the standard ESPRIT. An alternative way to compute ω_0 from $\mathbf{S}_2, \mathbf{S}_1$ is by the use of (5). More exactly, one has

$$\omega_0 = \arg\left(\sum_{k=1}^{p} z_k\right) = \arg\left(tr(\mathbf{D})\right)$$

$$= \arg\left(tr\left(\mathbf{S}_1^H \mathbf{S}_1\right)^{-1} \left(\mathbf{S}_1^H \mathbf{S}_2\right)\right) \qquad (12)$$

which leads to the alternative ESPRIT-based frequency estimate

$$\widehat{\omega}_0^t = \arg(tr[(\widehat{\mathbf{S}}_1^H \widehat{\mathbf{S}}_2)^{-1}(\widehat{\mathbf{S}}_1^H \widehat{\mathbf{S}}_2)]) \qquad (13)$$

2.2. MODE

The main idea behind the MODE method [4] is to reparametrize the noise subspace in terms of a pth degree polynomial whose roots yield the $ARMA$ pole frequencies. More exactly, let:

$$C(z) = \sum_{k=0}^{p} c_k z^{p-k} = c_0 \prod_{k=1}^{p} (z - z_k) \qquad (14)$$

Therefore, according to (4) and (5), ω_0 can be retrieved simply as

$$\omega_0 = \frac{1}{p} \arg((-1)^p \times c_p/c_0) \qquad (15)$$

$$= \arg(-c_1/c_0) \qquad (16)$$

Let us define

$$\mathbf{B}^H = \begin{pmatrix} c_p & c_{p-1} & . & c_0 & & 0 \\ & c_p & & & . & \\ & & . & & & . \\ & & & c_p & & c_0 \end{pmatrix} \qquad (17)$$

We first note that $rank(\mathbf{B}) = m - p$. Moreover, using the Yule-Walker equations (3), it can be easily checked that

$$\mathbf{B}^H \mathbf{R} = 0 \Rightarrow \mathbf{B}^H \mathbf{S} = 0 \qquad (18)$$

which implies that $\mathbf{B}$ spans the noise subspace of $\mathbf{R}$. The MODE algorithm estimates the coefficients $\{c_k\}$ by minimizing the following cost function

$$f(\mathbf{c}) = tr\left[\mathbf{B}\mathbf{B}^H \widehat{\mathbf{S}} \mathbf{W} \widehat{\mathbf{S}}^H\right] \qquad (19)$$

where $\mathbf{W}$ is a positive definite weighting matrix. Since the singular vectors in $\widehat{\mathbf{S}}$ are usually estimated with

an accuracy that is proportional to the corresponding singular values, a good choice for $\mathbf{W}$ is given by $\mathbf{W} = diag(\widehat{\sigma}_1, ..., \widehat{\sigma}_p) = \widehat{\mathbf{\Sigma}}$. $f(\mathbf{c})$ in (19) can be rewritten in the following form, more amenable to numerical minimization:

$$f(\mathbf{c}) = \mathbf{c}^H \widehat{\Omega} \mathbf{c} \tag{20}$$

where $\widehat{\Omega}$ is a $p \times p$ matrix which can be easily constructed from $\mathbf{W}$ and $\widehat{\mathbf{S}}$. The previous function must be minimized with an appropriate constraint. Here, we impose $\|\mathbf{c}\| = 1$, so that the solution is given by the smallest eigenvector of $\widehat{\Omega}$. Once an estimate $\widehat{\mathbf{c}}$ of $\mathbf{c}$ is available, the two forms of frequency estimate discussed before are implemented as:

$$\widehat{\omega}_0^d = \frac{1}{p} \arg((-1)^p \times \widehat{c}_p/\widehat{c}_0) \tag{21}$$

$$\widehat{\omega}_0^t = \arg(-\widehat{c}_1/\widehat{c}_0) \tag{22}$$

3. NUMERICAL ILLUSTRATIONS

In this section, the performance of ESPRIT and MODE frequency estimators is first examined via Monte-Carlo simulations. Secondly, an application to real radar data is presented.

3.1. Monte Carlo simulations

In this section, we illustrate and contrast the respective performances of the four estimators (11), (13), (21) and (22). In what follows, the envelope is selected as an $AR(2)$ process with poles at $\rho e^{\pm j2\pi f}$ and the variance of the driving noise is $\sigma^2 = 0.1$. We successively investigate the performance of the frequency estimates as a function of N (cf. Fig. 1) and the lowpass envelope parameter f (see Fig. 2). In all simulations, the sinusoid frequency is selected as $\omega_0 = 2\pi \times 0.18$ and 500 Monte-Carlo simulations are run to estimate the variances of the estimated frequency. The additive noise variance is chosen as $\sigma_w = 0.01$.

It can be seen that the determinant and trace method are practically equivalent. Additionally, MODE is observed to perform slightly better than ESPRIT. Figure 2 also reveals that the estimation performances remain quite stable over a wide range of frequencies.

3.2. Application to radar data

In this section we consider the problem of estimating the speed of trains from an on-board Doppler radar. So far, trains' speed is measured by means of a wheel which delivers impulses each time a certain distance is run. In most of the trains, the wheel is not equipped with an "anti-skid system" which means that when the
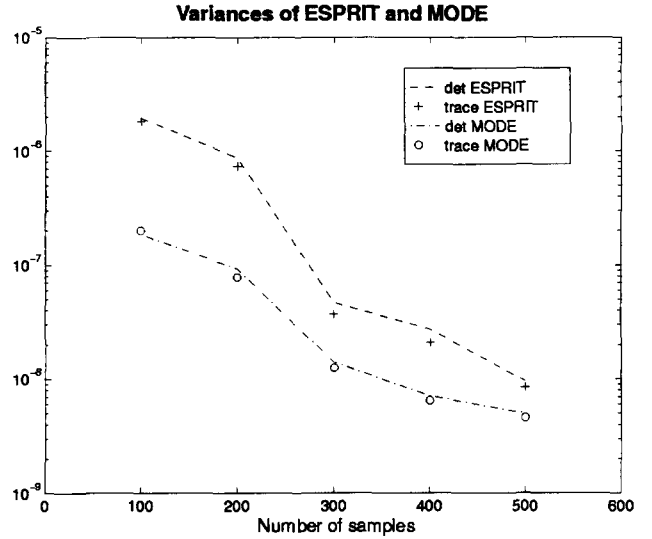

Figure 1: Variance of ESPRIT and MODE versus N. $M = 15$, $m = 10$, $\rho = 0.95$ and $f = 0.01$

train brakes or accelerates, the wheel is spinning resulting in a loss of accuracy. To remedy this problem, an on-board Doppler radar sends a continuous wave towards tracks and receives the echoes. After demodulation, the signal consists of a sinusoidal signal (whose frequency can be used to determine the train's speed) with a slowly fluctuating envelope. An important drawback of parametric methods when applied to real data is that their performance may seriously degrade when the model does not perfectly fit the received data. Therefore, designing robust algorithms is a key issue. We now propose a scheme that could "robustify" our algorithm without penalizing too much of its performance. We propose to apply the frequency estimation algorithms with an increasing number of "signal" singular vectors, until a "breakdown" is observed in the value of the estimated frequency. If a singular vector which does not contain information about the signal is included in the algorithm, this will typically result in a frequency estimate which is not plausible in the sense that it is out of a predefined range (which can be determined from the current train's speed and its maximum acceleration). We now formalize the idea above. From the sample covariance matrix $\widehat{\mathbf{R}}$, we form a complete orthogonal decomposition of it (a QRD with column pivoting is used in what follows) which provides a basis for the span of $\widehat{\mathbf{R}}$ $[\widehat{\mathbf{u}}_1, ..., \widehat{\mathbf{u}}_m]$. Then the scheme in Table 1 is applied.

If the envelope varies very slowly, then selecting only one singular vector will provide an accurate estimate (see [5]). The inclusion of additional singular
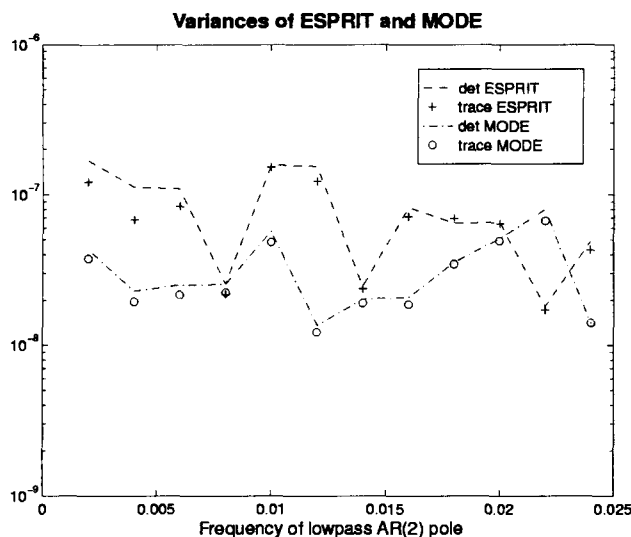
Variances of ESPRIT and MODE

Figure 2: Variance of ESPRIT and MODE versus f. $N = 300$. $M = 20$, $m = 10$ and $\rho = 0.95$.

1: $k = 1$
2: Define $\widehat{\mathbf{S}}^k = [\widehat{\mathbf{u}}_1, ..., \widehat{\mathbf{u}}_k]$
3: Compute $\widehat{\omega}_0^k$ from $\widehat{\mathbf{S}}^k$.
4: if $\widehat{\omega}_0^k$ is "plausible" go to 2 with $k = k + 1$, otherwise exit with $\widehat{\omega}_0^{k-1}$.

Table 1: Robustified scheme for estimating ω_0.

vectors will provide a plausible frequency as long as the envelope varies sufficiently rapidly for being modeled by a sum of k damped harmonics. We illustrate the behavior of this "robustified" algorithm when applied to real radar data recorded on-board a train. In the experiments described below, a non-skidding wheel provided information about the distance run by the train. More exactly, each 10 meters an impulse is delivered to indicate the distance. This provides a reference against which the estimate obtained with our algorithm is compared. The radar data was divided into successive (non-overlapping) blocks of $N = 512$ points; the sampling frequency was equal to $16kHz$. The parameters were selected as $M = 30$ and $m = 10$. The distance is then estimated by integrating over time the estimated speed, assuming that the speed is constant over a block of N samples. Each time a reference impulse is delivered, the estimated distance is compared with the true value, i.e. 10 meters. In case 1, a total number of 13460 meters were run, 4560 in 2nd case. Tables 2,3 list the results obtained, for each case, using either the ESPRIT or the MODE estimator. In these tables, $\overline{d}$ denotes the mean value of the distance estimate, std its standard deviation. The two last lines indicate the percentage of times the dimension of the signal subspace (SS) was selected as one or two.

	ESPRIT	MODE
$\overline{d}$	9.9069	9.923
std	0.088	0.083
$\dim(SS) = 1$	37.5%	59.8%
$\dim(SS) = 2$	62.5%	40.2%

Table 2: Results obtained in case 1

	ESPRIT	MODE
$\overline{d}$	10.005	9.988
std	0.0562	0.088
$\dim(SS) = 1$	54.8%	58.1%
$\dim(SS) = 2$	45.2%	41.9%

Table 3: Results obtained in case 2

As can be seen from these tables, the two estimators provide an accurate value of the true distance. Additionally, they give estimates close to each other, which confirms the observations made on simulated data. Moreover, we note that, in a non negligible way, two vectors were used to estimate the Doppler frequency, which proves the usefulness of varying the dimension of signal subspace to take into account the fluctuating amplitude.

4. REFERENCES

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