Total Least Squares Linear Prediction for Frequency Estimation with Frequency Weighting

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Abstract

This paper presents a general total least squares (GTLS) solution for linear prediction to estimate closely spaced sinusoids. It is found that the TLS prediction error is not a good criterion to provide a robust solution. In this paper, a frequency weighted prediction error approach is introduced. Experimental results show that the GTLS solution based on the frequency weighted prediction error can give a robust performance even in very low SNR.

I. Introduction

Resolving closely spaced sinusoids in the presence of noise is a very difficult problem particularly when the number of data is small and the SNR is low. Various methods have been developed in solving this problem such as autoregressive (AR) and autoregressive moving average (ARMA) modeling [1], Pisarenko method [2], principle eigenvector (PE) approach [3] and the total least squares (TLS) approach [4,5,6]. It has been stated that PE can be considered as a generalization of least squares (LS) solution and TLS of Pisarenko method. When both SNR is low and data length is short, TLS method provides better results than PE method.

In this paper, a general TLS solution (GTLS) is presented. It can be proved that the minimum norm TLS solution (MTLS) and the TLS solution developed by Rahman and Yu [4] (RYTLS) are the special cases of the general solution. RYTLS is a TLS solution based on the PE approximation of the data matrix and the observation vector whereas the GTLS to be discussed in this paper is a general solution for the PE approximation.

In view of the results obtained from MTLS, the TLS prediction error is definitely not a good criterion to provide a robust solution particularly for low SNR. To overcome this problem, we use a frequency weighting filter to pre-emphasize the data in the signal band. This is equivalent to emphasize the frequency weighting of prediction error in the signal band. Applying the frequency weighted data to the TLS method, the resolution and the variance of the frequency estimates are significantly improved. It is shown that the general solution together with the frequency weighted prediction error criterion can give a robust performance even in very low SNR.

II. Frequency Estimation and Total Least Squares Problem

The model of the received signal y_n is described as:

$$y_n = \sum_{k=1}^{K} a_k \cos(\omega_k n + \theta_k) + w_n \ , \quad n = 0,1,...N-1 \qquad (1)$$

where a_k, ω_k and θ_k are the amplitude, frequency and phase of the k-th sinusoid, respectively. The noise samples $\{w_n\}$ are assumed to be Gaussian distributed. The frequency estimation problem is to estimate the frequencies and amplitudes of the K sinusoids from the received data record. The following set of linear equation is commonly used to solve this problem.

$$\begin{bmatrix} y_0 & y_1 & \cdots & y_{p-1} \\ y_1 & y_2 & \cdots & y_p \\ \vdots & \vdots & \ddots & \vdots \\ y_{N-p-1} & y_{N-p} & \cdots & y_{N-2} \end{bmatrix} \begin{bmatrix} x_p \\ x_{p-1} \\ \vdots \\ x_1 \end{bmatrix} \approx \begin{bmatrix} y_p \\ y_{p+1} \\ \vdots \\ y_{N-1} \end{bmatrix}$$

$$\text{or } Ax \approx b \qquad (2)$$

where A is the linear prediction (LP) data matrix, x is the LP vector, and b is the observation vector. In general the order p of the LP vector x is larger than K.

Since the data are corrupted by noise, both the data matrix and the observation vector are contaminated. The TLS approach is more appropriate for solving the LP problem. The TLS problem is basically defined as follows:

Given an overdetermined set of linear equations $Ax \approx b$ with p unknowns x, where A is a $N \times p$ matrix and b is $N \times 1$ vector, the TLS problem seeks to

$$\underset{[\hat{A},\hat{b}] \in R^{N \times (p+1)}}{\text{minimize}} \left\| [A,b] - [\hat{A},\hat{b}] \right\|_F \tag{3}$$

subject to $\hat{b} \in$ range $(\hat{A})$.

Once a minimized $[\hat{A},\hat{b}]$ is found, then any x satisfying $\hat{A}x = \hat{b}$ or $[\hat{A},\hat{b}]\begin{bmatrix} x \\ -1 \end{bmatrix} = 0$ is called a TLS solution.

Applying singular value decomposition (SVD) to A and $[A,b]$ with $\{\sigma'_n\}$ and $\{\sigma_n\}$ denoting the singular values of A and $[A,b]$, respectively, the TLS solution is given in (4) when $\sigma'_p > \sigma_{p+1}$.

$$x_{MTLS} = \frac{-1}{v_{p+1,p+1}} \begin{bmatrix} v_{p+1,1} \\ \vdots \\ v_{p+1,p} \end{bmatrix} \quad \text{provided } v_{p+1,p+1} \neq 0 \tag{4}$$

where $[v_{p+1,1} \ v_{p+1,2} \cdots v_{p+1,p+1}]^T$ is the $(p+1)$-th singular vector corresponding to the singular value σ_{p+1} of $[A,b]$.

Rahman and Yu [4] described the following algorithm based on PE approximation for computing the TLS solution.

1. Compute the SVD of $[A,b]$, i.e. $[A,b] = U \, \text{diag}\{\sigma_1, \cdots, \sigma_{p+1}\} V$.

2. Determine the index M by $\sigma_M > \sigma_{p+1} + \varepsilon \geq \sigma_{M+1} \geq \cdots \geq \sigma_{p+1}$ for some predefined $\varepsilon > 0$.

3. The TLS solution is given by

$$x_{RYTLS} = -\sum_{i=M+1}^{p+1} \frac{v_{i,p+1}}{\sum_{k=M+1}^{p+1} |v_{k,p+1}|^2} v'_i \tag{5}$$

where $v_i = \begin{bmatrix} v'_i \\ v_{i,p+1} \end{bmatrix}$.

In the case when $\sum_{i=M+1}^{p+1} |v_{i,p+1}|^2$ is too small, M can be reduced until a nonzero $v_{M+1,p+1}$ is reached.

As a comparison, it is shown from the experiments in section V that the results of x_{MTLS} is generally not as good as those of x_{RYTLS}.

III. General TLS Solution

The $[A,b]$ can be written into the following form using singular value decomposition:

$$[A,b] = \sum_{i=1}^{p+1} \sigma_i u_i v_i^+ \tag{6}$$

where u_i is a $N \times 1$ vector and v_i is a $(p+1) \times 1$ vector, and $^+$ denotes the complex conjugate matrix transpose. The principal singular vector approximation of $[A,b]$ using the largest M singular values is described by

$$[\hat{A},\hat{b}] = \sum_{i=1}^{M} \sigma_i u_i v_i^+ \tag{7}$$

Then the TLS solution for $[\hat{A},\hat{b}]$ is spanned by the set of singular vectors $[v_{M+1}, \cdots, v_{p+1}]$. Mathematically, the TLS solution is given by

$$x_{GTLS} = -\sum_{i=M+1}^{p+1} \frac{\beta_i v_{i,p+1}}{\sum_{k=M+1}^{p+1} |\beta_k v_{k,p+1}|^2} v'_i \tag{8}$$

where $\{\beta_i\}$ are some positive real numbers.

Let β_i be related to the singular value σ_i as follow:

$$\beta_i = \sigma_i^{-q} \tag{9}$$

Then, the general TLS solution is given by

$$x_{GTLS} = -\sum_{i=M+1}^{p+1} \frac{\sigma_i^{-q} v_{i,p+1}}{\sum_{k=M+1}^{p+1} |\sigma_k^{-q} v_{k,p+1}|^2} v'_i \tag{10}$$

When $q \to \infty$, x_{GTLS} becomes x_{MTLS}; while $q = 0$, $x_{GTLS} = x_{RYTLS}$.

According to the GTLS solution in (10), the prediction error e is given by

$$e = \left[\sum_{i=M+1}^{p+1} \sigma_i u_i v_i^+ \right] \begin{bmatrix} x_{GTLS} \\ -1 \end{bmatrix} \tag{11}$$

The mean squared prediction error is therefore given by

$$E_{GTLS} = \sum_{i=M+1}^{p+1} \sigma_i^2 \left(v_i^+ \begin{bmatrix} x_{GTLS} \\ -1 \end{bmatrix} \right)^2$$

$$= \frac{\displaystyle\sum_{i=M+1}^{p+1} \sigma_i^{2(1-q)} |v_{i,p+1}|^2}{\left[\displaystyle\sum_{i=M+1}^{p+1} \sigma_i^{-q} |v_{i,p+1}|^2 \right]^2} \qquad (12)$$

From equation (12), the mean squared prediction error of x_{MTLS} ($q = \infty$) is always smaller than that of x_{RYTLS} ($q = 0$). However, the results of RYTLS is generally much better than that of MTLS. Therefore the performance of TLS solution cannot be measured simply by evaluating the prediction error. In next section, we introduce a new criterion based on frequency weighting approach to provide a practical measure to give a robust GTLS solution.

IV. Frequency Weighted Total Least Squares Criterion

The approach is to pass the data $\{y_n\}$ to the frequency weighting filter described as

$$W(z) = \frac{1 - \displaystyle\sum_{i=1}^{p} a_{LS}(i)\gamma^i z^{-i}}{1 - \displaystyle\sum_{i=1}^{p} a_{LS}(i) z^{-i}} \qquad (13)$$

where $[a_{LS}(1), \cdots, a_{LS}(p)]^T$ is the least squares solution to equation (2). Let $\{z_n\}$ denote the output from the filter $W(z)$. The purpose of using $W(z)$ is to pre-emphasize the signal in $\{z_n\}$ to improve the effective SNR. It is shown that the parameter γ in the range of 0.2 to 0.6 can in general provide good results.

The outputs $\{z_n\}$ are then applied to equation (2) and the GTLS solution is given by (10). Applying the frequency weighting filter to the data $\{y_n\}$ is equivalent to weight the error more heavily in the signal band. Therefore the TLS problem becomes minimizing frequency weighted prediction error. It is shown in section V that the GTLS solution using frequency weighting provides a significant performance gain over MTLS and RYTLS.

V. Experimental Results

The signal model of the experiments is defined in (1). We consider two closely spaced sinusoids with frequencies $\omega_1 = 0.7813\pi$ and $\omega_2 = 0.7617\pi$ and amplitudes equal. The phases of the sinusoids are randomly selected from $(0, 2\pi)$. The noise $\{w_n\}$ is Gaussian distributed. Two SNRs are considered (0.5 dB and 5 dB) in the experiments. The length N is set equal to 128. The following results are averaged over 100 runs. For RYTLS and GTLS, the number of principal eigenvectors, M, is set equal to 4.

In Fig.1, the frequency responses of MTLS and RYTLS are compared for SNR = 5 dB. It is clear that the result of RYTLS is much better than that of MTLS. It verifies that minimum norm criterion does not necessarily provide a good solution. Fig. 2 shows the *unsuccessful percentage* of resolving the two sinusoids. The criteria for determining the unsuccessful cases are: (i) the radius of the roots of the LP polynomial is less than 0.9; and (ii) the angles of the roots are outside the range of $\omega_2 - \Delta\omega$ and $\omega_1 + \Delta\omega$ where $\Delta\omega = \omega_1 - \omega_2$. For the cases of without weighting filter, more than 60% and 85% of most GTLS solutions (of different q) are unresolved for 5 dB and 0.5 dB respectively. Comparing the cases with weighting filter, the successful percentage is substantially improved. And in particular, all cases can be resolved for 5 dB when $-2.75 \le q \le 5.5$. These results demonstrate the effectiveness of the weighting filter.

To illustrate the frequency estimation error of using the GTLS with weighting filter, the mean squared frequency error (MSFE) defined as

$$MSFE = 10 \log_{10} \frac{1}{KL} \sum_{l=1}^{L} \sum_{i=1}^{K} (\hat{\omega}_i^{(l)} - \omega_i)^2 \qquad (14)$$

where L equals the number of successful resolvable cases. Fig. 3 shows the MSFE for SNR = 5 dB with $\gamma = 0.5$. The frequency estimate $\hat{\omega}_i^{(l)}$ is calculated from the roots of the LP polynomial. Finally, the frequency response of GTLS using frequency weighting with $q = 0$ and $\gamma = 0.5$ for 5 dB is plotted in Fig. 4. Comparing with the results in Fig.1, GTLS with weighting filter provides a very good solution with small variance.

Conclusions

A general TLS solution is discussed in this paper. By applying frequency weighting to the linear prediction, a robust TLS solution is obtained and its performance is almost insensitive to SNR even close to 0 dB and a small mean squared frequency error is achieved.

Acknowledgment

Reference

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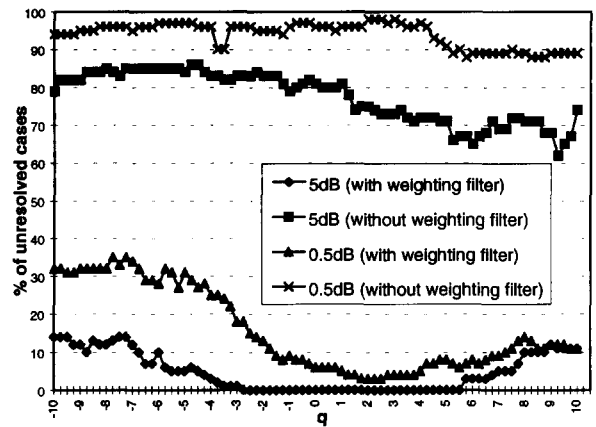
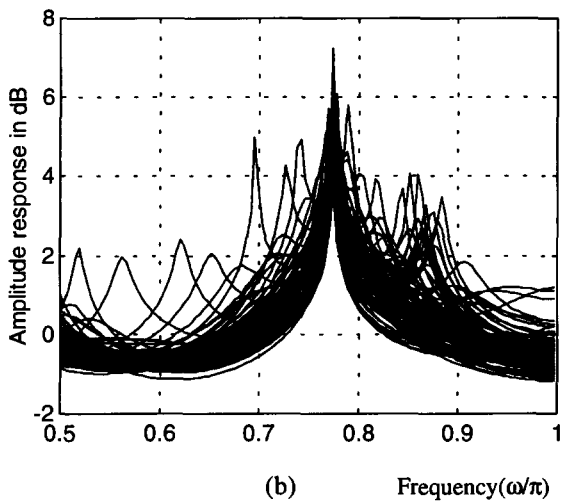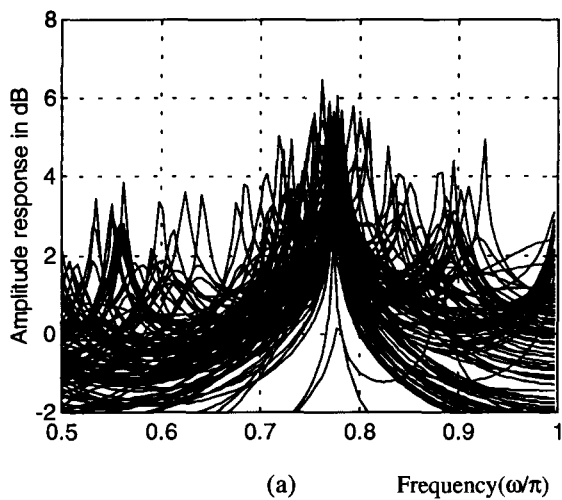Fig.2: % of unresolved cases for SNR=5dB and 0.5dB using GTLS with and without weighting filter

(a)   Frequency(ω/π)

Fig.3: Mean squared frequency error of all resolved cases for 5dB (i.e. $-2.75 \leq q \leq 5.5$)

(b)   Frequency(ω/π)

Fig.1: Frequency responses of MTLS and RYTLS for SNR=5dB (a) MTLS, (b) RYTLS
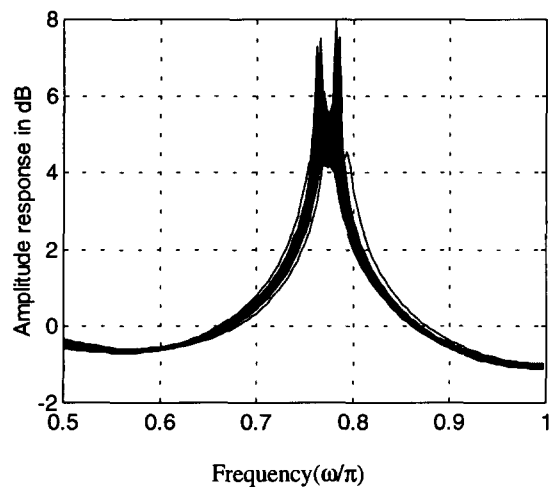

Frequency(ω/π)

Fig.4: Frequency responses of GTLS with frequency weighting (q=0 and γ=0.5) for SNR=5dB