Optimal Cyclic Statistics for the Estimation of Harmonics in Multiplicative and Additive Noise

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Abstract— **The problem of detection and estimation of harmonics in multiplicative and additive noise is addressed. The problem may be solved using *i*) the cyclic mean if the harmonic amplitude is not zero mean or *ii*) the cyclic variance if the harmonic amplitude is zero mean. Solution *ii*) may be used when the amplitude of the harmonic is not zero mean while solution *i*) fails in the case of zero mean harmonic amplitude. This paper answers the following question: given a multiplicative and additive noisy environment, which solution is optimal? The paper determines thresholds on the coherent to non coherent sine powers ratio which delimitate the regions of optimality of the two solutions. Comparison with higher-order cyclic statistics is presented. Gaussian as well as non Gaussian noise sources are studied.**

I. Introduction

The problem of harmonics retrieval has received much attention in the literature. Most of the existing methods focus on extracting coherent harmonics from additive noisy data. However, multiplicative noise is encountered in several signal processing problems. For example, random amplitude modulation phenomenon occurs in propagating underwater signals due to the medium [2] and in radar signals due to scintillations of the targets [8][1]. In this paper, we will be concerned with detection and estimation of a sinusoidal signal which has corrupted by both multiplicative and additive noise. The multiplicative noise decorrelates the sinusoid, spreads its power spectrum and acts as an additional corrupting noise.

Consider N measurements of a discrete-time sinusoid in additive and multiplicative noise

$$x(t) = (\mu + y(t))\cos(\omega t + \phi) + \nu(t), \quad t = 0, ..., N-1 \qquad (1)$$

with the following assumptions: $A1$) the deterministic amplitude $\mu \geq 0$; $A2$) $y(t)$ is a white multiplicative noise with variance σ_y^2; $A3$) ω and ϕ are the frequency and the phase of the sinusoidal signal, assumed constants in the ranges $(0, \pi/2)$ and $(-\pi, \pi]$ respectively; $A4$) $\nu(t)$ is a white additive noise with variance σ_ν^2; $A5$) the multiplicative and additive noise sources are mutually independent.

In the classical case of a constant amplitude harmonic in additive noise, i.e. $\mu \neq 0$ and $\sigma_\nu^2 = 0$, the signal-to-noise ratio (SNR) relative to the coherent sinusoid in the signal x_t^2 is at most an eighth that of x_t as will be shown in section III. This explains why the spectrum of x_t^2 is never considered in practice. The aim of this paper is to reconsider this result when the harmonic is corrupted by a multiplicative noise.

Alternatively, if $\mu = 0$, the spectral analysis of x_t contains in general no information about the sinusoidal signal parameters. However, the spectral analysis of x_t^2 enables the detection and estimation of the harmonic provided $\sigma_y^2 \neq 0$. This is the basic idea of the Cyclic Variance (CV) [9][6] and fourth-order cumulant based methods [2][7]. Nevertheless, the spectral analysis of $x(t)$ is useful if the multiplicative noise is a narrow band signal. For instance, the frequency can be estimated using the center of gravity of the spectrum of $x(t)$ [1][4].

In the general case where $\mu \neq 0$ and $\sigma_y^2 \neq 0$, [9] and [6] advocate the use of the Cyclic Mean (CM). However, the CV method enables to solve the harmonic retrieval problem too. One must therefore wonder which statistic yields the better performance. It is worth noting that the CM and the CV techniques are spectral analysis of x_t and x_t^2 respectively. In this paper, we will demonstrate that the CV method outperforms the CM method not only for $\mu = 0$, but also for non zero values of μ ranging from 0 to a threshold which depends on the probability density function (pdf) of the noise sources. In other words, the paper answers the following question: for given μ and σ_ν^2, up to which value of the multiplicative noise power σ_y^2, is the CM method optimal?

II. Harmonic Retrieval using the CM and CV Statistics

The beginning of this section presents the cyclic approach for the harmonic parameter estimation proposed in [9]. We then compute the finite sample variances of the CM and the CV statistics. This will enable us to derive the optimality criterion proposed in the next section.

The periodic nature of the sinusoidal signal makes $x(t)$ cyclostationary [3]. Since the mean and the variance of $x(t)$ are periodically time-varying, one considers their generalized Fourier series coefficients, which are called the cyclic mean (CM) and the cyclic variance [3]. The frequency ω can be estimated by [9]

$$\widehat{\omega}^{(1)} = \arg\max_{\alpha>0} \left|\widehat{M}_{1x}(\alpha)\right| \quad \text{or} \quad \widehat{\omega}^{(2)} = \frac{1}{2}\arg\max_{\alpha>0} \left|\widehat{M}_{2x}(\alpha;0)\right| \qquad (2)$$

where $\widehat{M}_{1x}(\alpha) = \frac{1}{N}\sum_{t=0}^{N-1} x_t e^{-j\alpha t}$ and $\widehat{M}_{2x}(\alpha;0) = \frac{1}{N}\sum_{t=0}^{N-1} x_t^2 e^{-j\alpha t}$. The CV method requires ω to be in $(0, \pi/2)$ in order to avoid the aliasing phenomenon. In order to carry out the performance evaluation of the CM and CV methods, one needs the following proposition. Let m_{3y}, m_{4y}, $m_{3\nu}$ and $m_{4\nu}$ be the third and fourth-order moments of y_t and ν_t respectively.

Proposition. *The finite sample variances of the CM and the CV estimates are, under assumptions A1-A5, given by*

$$\begin{aligned}
var\left\{\widehat{M}_{1x}(\alpha)\right\} &= \tfrac{1}{N}\left(\tfrac{\sigma_y^2}{2} + \sigma_\nu^2\right) + \tfrac{1}{2N^2}C_N(2\omega, 2\phi) \\
var\left\{\widehat{M}_{2x}(\alpha;0)\right\} &= \tfrac{1}{N}\left(4\mu^2\sigma_y^2 + 4\mu m_{3y} + m_{4y} - \sigma_y^4\right) \\
&\quad \left(\tfrac{3}{8} + \tfrac{1}{2N}C_N(2\omega, 2\phi) + \tfrac{1}{8N}C_N(4\omega, 4\phi)\right) \\
&\quad + \tfrac{1}{N}\sigma_\nu^2\left(\mu^2 + \sigma_y^2\right)\left(2 + \tfrac{2}{N}C_N(2\omega, 2\phi)\right) \\
&\quad + \tfrac{1}{N}\left(m_{4\nu} - \sigma_\nu^4\right) + \tfrac{1}{N^2}4\mu m_{3\nu}C_N(\omega, \phi)
\end{aligned}$$

where

$$C_N(k\omega, k\phi) \triangleq \frac{\sin(\frac{k\omega}{2}N)}{\sin(\frac{k\omega}{2})}\cos(\tfrac{k\omega}{2}(N-1)+k\phi),\ \forall k \in \mathbf{R}$$

The above proposition provides the exact variances of the CM and CV statistics for any number of measurements. The terms in N^{-2} are functions of the *Dirichlet Kernel* $\sin(\lambda N)/\sin(\lambda)$, which is equal to N when λ is an integer multiple of π. The terms in N^{-2} can then be neglected if ω and 2ω are outside the main lobes of the *Dirichlet Kernel* centered at 0 and π. Therefore, in what follows, N will be assumed large enough to satisfy

$$N > \max\left\{\frac{\pi}{\omega}, \frac{\pi/2}{\pi/2-\omega}\right\} \tag{3}$$

III. CM versus CV

According to section II, both the CM and the CV method may be used for the detection and estimation of the sinusoidal signal if $\mu \neq 0$. The goal of this section is to provide statistical tools for choosing between the two methods. The criterion of optimality can be based on the measure of the ratios

$$\mathrm{R}_1 = \frac{\left|E\left\{\widehat{M}_{1x}(\omega)\right\}\right|}{\sqrt{var\left\{\widehat{M}_{1x}\right\}}};\quad \mathrm{R}_2 = \frac{\left|E\left\{\widehat{M}_{2x}(2\omega;0)\right\}\right|}{\sqrt{var\left\{\widehat{M}_{2x}\right\}}} \tag{4}$$

The greater these ratios, the better enhancement of the peak in the CM and CV statistics. The CV method is optimal if $\mathrm{R}_2 > \mathrm{R}_1$. The CM and the CV methods may simply be regarded as the estimation of a sinusoidal signal in additive noise with different SNRs. The more the SNR decreases, the more difficult it will be to detect the desired line spectrum. Next, we establish the link between R_1 and R_2 and the corresponding SNRs. Evaluating the performance by means of SNRs will enable us to generalize our results to any linear processor-based methods.

The signal x_t can be written as

$$x_t = s_1(t) + \xi_1(t) \tag{5}$$

where $s_1(t) = \mu\cos(\omega t + \phi)$ and $\xi_1(t) = y(t)\cos(\omega t + \phi) + \nu(t)$. Moreover, $s_1(t)$ and $\xi_1(t)$ are decorrelated. The CM-based harmonic retrieval regards $s_1(t)$ as the desired signal and $\xi_1(t)$ as an additive noise. We can now define the SNR which is relative to the CM method, under condition (3), as

$$SNR_1 \triangleq \frac{\frac{1}{N}\sum_{t=0}^{N-1} s_1^2(t)}{\frac{1}{N}\sum_{t=0}^{N-1} E\{\xi_1^2(t)\}} \simeq \frac{\mu^2/2}{\sigma_y^2/2+\sigma_\nu^2} \tag{6}$$

The index $_1$ refers to the order of the used cyclic statistic. In the same way, the signal and noise decomposition of x_t^2 results in a coherent sinusoid of frequency 2ω and a white additive noise. The SNR corresponding to the CV method under condition (3), is found to be

$$SNR_2 \simeq \frac{\frac{1}{8}(\mu^2+\sigma_y^2)^2}{\frac{3}{8}\left(4\mu^2\sigma_y^2+4\mu m_{3y}+m_{4y}-\sigma_y^4\right)+2\sigma_\nu^2(\mu^2+\sigma_y^2)+m_{4\nu}-\sigma_\nu^4} \tag{7}$$

Under condition (3), the ratios R_1 and R_2 and SNR_1 and SNR_2 are linked by

$$SNR_1 \simeq 2N\,\mathrm{R}_1^2; \qquad SNR_2 \simeq 2N\,\mathrm{R}_2^2 \tag{8}$$

We can then define the relative efficiency of the CV method with respect to the CM method as

$$RE \triangleq \frac{SNR_2}{SNR_1} \tag{9}$$

Before treating the multiplicative noise effects, it is worth noting that in the pure additive noise case (i.e. $\sigma_y^2 = 0$), RE is at most equal to 1/8:

$$RE = \left(8 + 4\left(\frac{m_{4\nu}}{\sigma_\nu^4} - 1\right)\frac{\sigma_\nu^2}{\mu^2}\right)^{-1} < 1/8 \tag{10}$$

since the ratio $\frac{m_{4\nu}}{\sigma_\nu^4}$ for continuous valued random variables is necessarily strictly greater than 1 due to Cauchy-Swharz inequality. Therefore, the CM is by far the best method in this case. In what follows, we will prove that this result can be wrong in the presence of multiplicative noise. Below, we generally limit our study to symmetrically distributed noise.

A-Multiplicative Noise Source.

Result 1. *In the case where the multiplicative noise is symmetrically distributed around* 0 *(the third order moment vanishes), the CV method is optimal if the coherent to non coherent sine powers ratio satisfies the following condition*

$$\frac{\mu^2}{\sigma_y^2} < \theta_1 = -\frac{3}{22}\frac{m_{4y}}{\sigma_y^4} + \frac{5}{22} + \frac{1}{22}\sqrt{\left(3\frac{m_{4y}}{\sigma_y^4} - 5\right)^2 + 44} \tag{C1}$$

Notice that the lower the ratio $\frac{m_{4y}}{\sigma_y^4}$, the greater the threshold θ_1 and the more the CV method becomes optimal. To study the relationship between the threshold θ_1 and the noise pdf, let us consider the generalized zero mean Gaussian pdf $f(y) = \frac{a}{2b\Gamma(1/a)}\exp - \left|\frac{y}{b}\right|^a$, where $\Gamma(.)$ is the gamma function, $a > 0$ is the shape parameter and $b > 0$ is the scale or size parameter. The threshold θ_1 as a function of a is plotted in figure 1. We then conclude that the more a increases (heavy-tailed pdfs), the greater the performance gain using the CV method.

B- Multiplicative and Additive Noise Sources. In this section, we study the influence of the additive noise on the above optimality results.

Result 2. *For symmetrically distributed multiplicative noise pdfs, the CV method is optimal if* $\frac{\mu^2}{\sigma_y^2}$ *satisfies*

$$\frac{\mu^2}{\sigma_y^2} < \theta_2 = \frac{-\gamma_2+\sqrt{\gamma_2^2-4\gamma_4\gamma_0}}{2\gamma_4} \tag{C2}$$

where $\gamma_4 = 14\frac{\sigma_\nu^2}{\sigma_y^2} + 11$, $\gamma_2 = 3\frac{m_{4y}}{\sigma_y^4} + 8\frac{\sigma_\nu^4}{\sigma_y^4}\left(\frac{m_{4\nu}}{\sigma_\nu^4} - 1\right) + 12\frac{\sigma_\nu^2}{\sigma_y^2} - 5$ and $\gamma_0 = -2\,\frac{\sigma_\nu^2}{\sigma_y^2} - 1$.

B-1- *Gaussian Multiplicative and Additive Noise sources.* Figure 2 displays the regions of optimality of the CM and the CV methods. Figure 2 shows that for a given value of σ_y^2, the domain of optimality of the CM method diminishes when the variance of the additive noise σ_ν^2 increases.

B-2-*Non Gaussian Multiplicative and Gaussian Additive Noise Sources.* We assume that the pdf of the multiplicative noise is the generalized Gaussian density. Thus, if the additive noise is Gaussian, the CM method is optimal if μ satisfies condition **(C2)** with $\gamma_2 = 3\Gamma(1/a)\Gamma(5/a)/\Gamma^2(3/a) + 16\sigma_\nu^4/\sigma_y^4 + 12\frac{\sigma_\nu^2}{\sigma_y^2} - 5$. Figure 3 displays the variations of the threshold θ_2 versus the ratio $\frac{\sigma_\nu^2}{\sigma_y^2}$ for different values of a.

IV. Harmonic Retrieval using Higher Order Cyclic Statistics

We have shown above that the square law transformation of $x(t)$ can improve greatly the estimation performance. Thus, the following question becomes imperative: what is the contribution of higher-order law transformations? In this paper, we limit our study to the third and fourth-order cyclic statistics.

A. Third-order cyclic statistics

The third-order cyclic moment (TCM) of $x(t)$, which is the generalized Fourier series coefficient of $m_{3x}(t;0,0)$, peaks at $\alpha = 0$, $\alpha = \omega$ and $\alpha = 3\omega$:

$$\begin{aligned} M_{3x}(\alpha;0,0) &\triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} m_{3x}(t;0,0)e^{-j\alpha t} \\ &= m_{3y}\delta(\alpha) + \frac{3}{8}\left[\mu^3 + 3\mu(\sigma_y^2 + \frac{4}{3}\sigma_\nu^2) + m_{3y}\right] \\ &\quad \left(e^{j\phi}\delta(\alpha-\omega) + e^{-j\phi}\delta(\alpha+\omega)\right) + \\ &\quad \frac{1}{8}(\mu^3 + 3\mu\sigma_y^2 + m_{3y})\left(e^{3j\phi}\delta(\alpha-3\omega) + e^{-j3\phi}\delta(\alpha+3\omega)\right) \end{aligned} \tag{11}$$

The TCM can be estimated consistently from a single realization by [3]

$$\widehat{M}_{3x}(\alpha;0,0) = \frac{1}{N} \sum_{t=0}^{N-1} x^3(t)e^{-j\alpha t} \tag{12}$$

A consistent estimator of ω is obtained by

$$\widehat{\omega} = \arg\max_{\alpha>0} \left|\widehat{M}_{3x}(\alpha;0,0)\right| \tag{13}$$

In contrast to the CV method, the TCM method has no limitation on the range of the harmonic frequency provided $\mu \neq 0$ or $m_{3y} \neq 0$. The TCM contains no information on the harmonic parameters if $\mu = 0$ in the case of symmetrically distributed multiplicative noise.

B. Fourth-order cyclic statistics

In the same manner as for the TCM, the fourth-order cyclic moment (FCM) of $x(t)$ is given by

$$\begin{aligned} M_{4x}(\alpha;0,0,0) &\triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} m_{4x}(t;0,0,0)e^{-j\alpha t} \\ &= \frac{1}{16}(\mu^4 + 6\mu^2\sigma_y^2 + 4\mu m_{3y} + m_{4y}) \\ &\quad \left(e^{4j\phi}\delta(\alpha-4\omega) + e^{-4j\phi}\delta(\alpha+4\omega)\right) \\ &\quad + \left[\frac{1}{4}(\mu^4 + 6\mu^2\sigma_y^2 + 4\mu m_{3y} + m_{4y}) + \frac{3}{2}(\mu^2 + \sigma_y^2)\sigma_\nu^2\right] \\ &\quad \left(e^{2j\phi}\delta(\alpha-2\omega) + e^{-2j\phi}\delta(\alpha+2\omega)\right) \\ &\quad + 2\mu m_{3y}\left(e^{j\phi}\delta(\alpha-\omega) + e^{-j\phi}\delta(\alpha+\omega)\right) + \\ &\quad \left[\frac{3}{8}(\mu^4 + 6\mu^2\sigma_y^2 + 4\mu m_{3y} + m_{4y}) + 3(\mu^2 + \sigma_y^2)\sigma_\nu^2 + m_{4\nu}\right]\delta(\alpha) \end{aligned} \tag{14}$$

Using $m_{3y}^2 < m_{4y}\sigma_y^2$ which results from Cauchy-Schwarz inequality, we get

$$|M_{4x}(2\omega;0,0,0)| > |M_{4x}(4\omega;0,0,0)| > |M_{4x}(\omega;0,0,0)| \tag{15}$$

Therefore, a consistent estimator of ω is obtained by

$$\widehat{\omega} = \frac{1}{2}\arg\max_{\alpha>0} \left|\widehat{M}_{4x}(\alpha;0,0,0)\right| \tag{16}$$

where $\widehat{M}_{4x}(\alpha;0,0,0) = \frac{1}{N}\sum_{t=0}^{N-1} x^4(t)e^{-j\alpha t}$.

V. CM and CV versus TCM and FCM

We limit the comparison between the four first cyclic statistics to the pure multiplicative noise case. Decomposing $x^3(t)$ and $x^4(t)$ into sinusoidal signals and noise components, the SNRs relative to the sinusoid of frequency ω in $x^3(t)$ and the sinusoid of frequency 2ω in $x^4(t)$ are found to be

$$\begin{aligned} SNR_3 &= \frac{\frac{9}{10}(\lambda^3 + 3\lambda\sigma_y^2 + \kappa_3)^2}{9\lambda^4 + 18\kappa_3\lambda^3 + \lambda^2(15\kappa_4 - 9) + 6\lambda(\kappa_5 - \kappa_3) + \kappa_6 - \kappa_3^2} \\ SNR_4 &= \frac{\frac{16}{35}(\lambda^4 + 6\lambda^2 + 4\kappa_3\lambda + \kappa_4)^2}{\left[(16\lambda^6 + 32\kappa_3\lambda^5 + 16(3\kappa_4 - 1)\lambda^4 + 8(5\kappa_5 - 4\kappa_3)\lambda^3 + 8(3\kappa_6 - \kappa_4 - 2\kappa_3^2)\lambda^2 + 8(\kappa_7 - \kappa_3\kappa_4)\lambda + \kappa_8 - \kappa_4^2)\right]} \end{aligned} \tag{17}$$

where

$$\lambda = \frac{\mu}{\sigma_y};\ \kappa_3 = \frac{m_{3y}}{\sigma_y^3};\ \kappa_4 = \frac{m_{4y}}{\sigma_y^4};\ \kappa_5 = \frac{m_{5y}}{\sigma_y^5};\ \kappa_6 = \frac{m_{6y}}{\sigma_y^6};\ \kappa_7 = \frac{m_{7y}}{\sigma_y^7};\ \kappa_8 = \frac{m_{8y}}{\sigma_y^8} \tag{18}$$

The In the case of symmetrically distributed multiplicative noise, simplified expressions of SNR_3 and SNR_4 are obtained by setting $\kappa_3 = 0$, $\kappa_5 = 0$ and $\kappa_7 = 0$. When the multiplicative noise pdf is the generalized Gaussian density, we obtain

$$\kappa_4 = \frac{\Gamma(\frac{1}{a})\Gamma(\frac{5}{a})}{\Gamma^2(\frac{3}{a})},\ \kappa_6 = \frac{\Gamma(\frac{1}{a})\Gamma(\frac{7}{a})}{\Gamma^2(\frac{3}{a})},\ \kappa_8 = \frac{\Gamma(\frac{1}{a})\Gamma(\frac{9}{a})}{\Gamma^2(\frac{3}{a})} \tag{19}$$

Figure 4 displays the variations of SNR_1, SNR_2, SNR_3 and SNR_4 versus the ratio μ^2/σ_y^2 for different values of the shape parameter a. It turns out that, for given values of μ^2/σ_y^2 and a, the maximum value of these SNRs is either SNR_1 or SNR_2. Therefore, the pair of methods CM and CV, is optimal with respect to the pair TCM and FCM, whatever the value of the shape parameter a.

VI. Simulations Results

To illustrate our theoretical results, we have computed the CM and the CV for a number of realizations of the signal given in (1). The sinusoidal frequency and phase are fixed at 0.4π and $\pi/3$ respectively for all the experiments. In this paper, we only present simulations in the pure Gaussian multiplicative noise case. The normalized (by the maximum over $\alpha > 0$) CM and CV for different value of the ratio $\frac{\mu^2}{\sigma_y^2}$ and $N = 128$ are depicted in figure 5. We note that the simulation results are in accordance with the theoretical predictions. The CV method provides superior enhancement of the line spectrum with respect to the CM method for small values of $\frac{\mu^2}{\sigma_y^2}$. The converse is true when $\frac{\mu^2}{\sigma_y^2}$ exceeds 0.1703.

VII. Conclusions

In this paper, we have reconsidered some commonly used results on the estimation of harmonics in a noisy environment. In contrast to the additive noise case, the spectral analysis of the square of the signal can result in increased estimation accuracy and power detection compared to the spectral analysis of the signal itself in the presence of multiplicative noise. Thus, the cyclic variance method is shown to be optimal with respect to the cyclic mean method in certain noisy environments. This is true not only for zero mean harmonic amplitude, but also for values of the mean ranging from 0 to a threshold which depends on the probability density functions of the noise sources. The CV method is shown to be more and more optimal for heavy-tailed multiplicative noise pdfs. It is also shown that the pair CM and CV methods outperforms methods based on higher-order cyclic moments for a large class of multiplicative noise pdfs including the Gaussian one. The choice of the optimal statistic is crucial for short data records lengths. The complex harmonic case will be presented elsewhere.

The optimal choice between the signal and its power-law transformations derived in the paper is valid for any linear processor. For example, the LMS adaptive line enhancer may give better performance when its input is the square of the signal rather than the signal itself [5]. This is due to the fact that the output SNR obtained at the convergence (i.e. the Wiener solution) is an increasing function of the input SNR. The following table summarizes our results (where θ_2 is defined in result 2)

Conditions	$\mu^2 > \theta_2\sigma_y^2$	$\mu^2 < \theta_2\sigma_y^2$
Optimal input to linear processors	$x(t)$	$x^2(t)$

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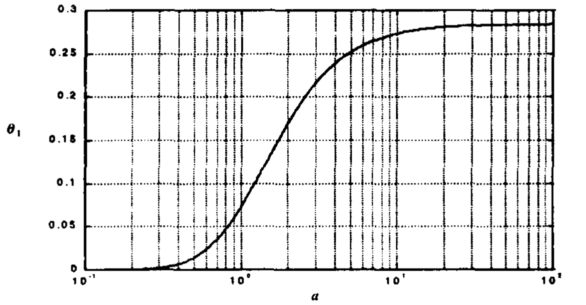

Fig. 1 : Threshold of optimality θ_1 versus the shape parameter a of the generalized Gaussian pdf.

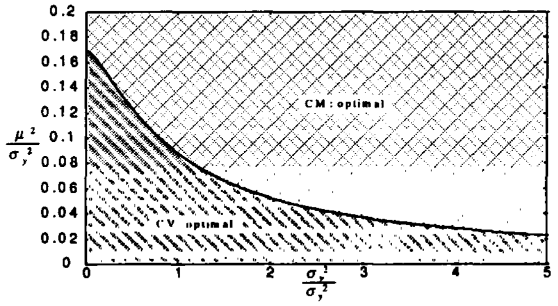

Fig. 2 : Regions of optimality of the CM and the CV methods in the case of Gaussian multiplicative and additive noise sources

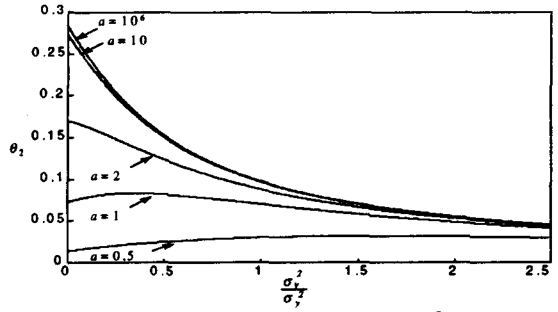

Fig. 3 : Threshold of optimality θ_2 versus $\frac{\sigma_x^2}{\sigma_y^2}$ in the case of generalized Gaussian multiplicative noise, with shape parameter a, and Gaussian additive noise.

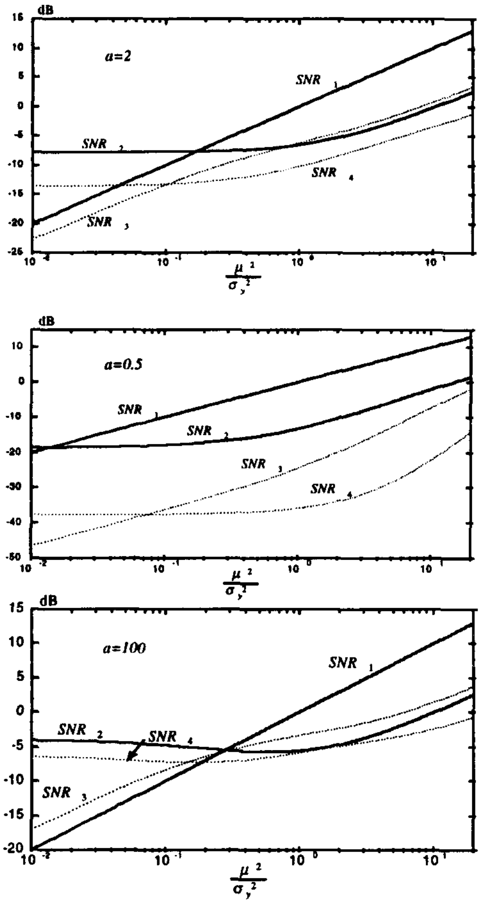

Fig. 4 : Plot of SNR_1, SNR_2, SNR_3 and SNR_4 versus $\frac{\mu^2}{\sigma_y^2}$ for different values of the shape parameter a.

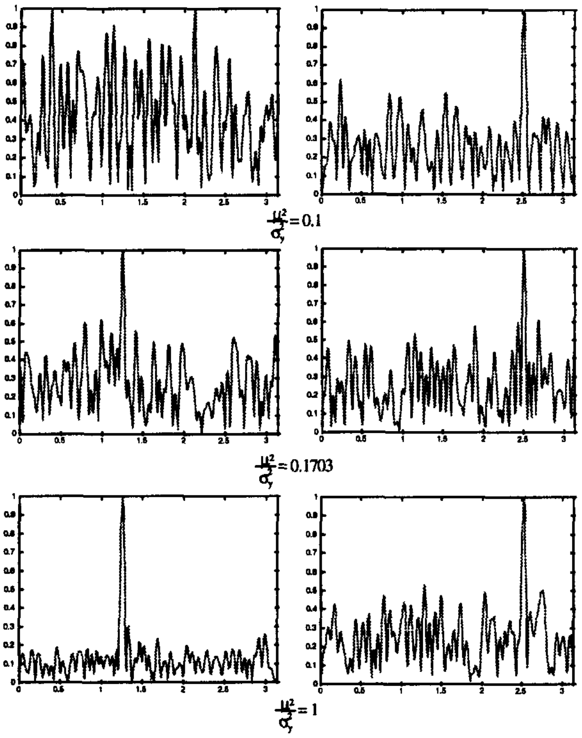

Fig. 5 : Normalized CM curves (left) and CV curves (right) for different values of $\frac{\mu^2}{\sigma_y^2}$ in the case of Gaussian multiplicative noise. $N = 128$.