

FREQUENCY ESTIMATION OF LASER SIGNALS WITH TIME-VARYING AMPLITUDE FROM PHASE MEASUREMENTS

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ABSTRACT

This paper addresses the problem of estimating particle's velocity in the vicinity of an aircraft by means of a laser velocimeter. A model for the signal generated by a particle of air passing through a probe volume consisting of equidistant bright and dark fringes is given. From this model, a frequency estimator based on the phase of the correlation sequence of the signal is proposed. A theoretical analysis of the frequency estimator is presented. In particular, a formula for the variance of the estimate is derived under the assumption of small estimation errors. Numerical examples confirm the validity of the analysis. Finally, the effectiveness of the proposed algorithm is demonstrated on real data.

1. OUTLINE

In aeronautics applications, there is a vital interest in having a reliable way of measuring aircraft's speed. Moreover, the size and weight of the system in charge of this task is of primary concern. So far, this problem has been solved with the help of a Pitot device which measures a difference between a static (*i.e.* perpendicular to the flow) and a dynamic pressure (*i.e.* parallel to the flow). This difference is directly related to the airplane's speed. Laser velocimeters have gained popularity in the fluid mechanics application, where they have been used to estimate particles velocity in a flow[1], mainly for measurements in wind tunnels. This paper addresses the problem of estimating particle's velocity in the vicinity of an aircraft by means of a laser velocimeter. To this aim, a symmetric interference fringe pattern composed of bright and dark fringes is generated in the vicinity of the aircraft by means of two coherent laser beams which are crossed and focused. A

particle of aerosol passing through this probe volume will alternatively encounter dark and bright fringes, therefore scattering light according to its velocity. In the noise free case, the signal recorded by a photodetector would consist of a sinusoidal signal (whose frequency is representative of particle's velocity, hence of aircraft's speed) with a time-varying amplitude of the form $A \cdot \exp \{-2\alpha^2 f_d^2 t^2\}$. In this paper, we address the problem of estimating the frequency of such a signal. In [2], we derived the Cramér-Rao Bound for the problem at hand and proposed a Maximum Likelihood Estimator of f_d . Although the MLE is statistically efficient, it requires the minimization of a non-linear function by means of iterative techniques. As it may be computationally too much intensive for real-time applications, an alternative approach is proposed here. Since the work of Tretter [3] who showed that, in the high Signal to Noise Ratio (*SNR*) scenario, an additive white noise on the signal was equivalent to an additive noise on the phase, estimators based on the phase of the signals have been proposed (see e.g. [4],[5]). These estimators are computationally efficient, yet possessing good statistical properties. In this paper, a frequency estimator based on the phase of the correlation sequence of the signal is proposed. It can be viewed as the generalization of these "phase-based" methods to an exponential signal with time-varying amplitude. A theoretical analysis enables to derive the variance of the frequency estimator, under the assumption of small estimation errors. Numerical examples confirm the validity of the analysis. It is reported that this estimator, although simpler than the Maximum Likelihood approach, comes close to the Cramér-Rao Bound. Finally, the effectiveness of the proposed algorithm is demonstrated on real data.

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2. FREQUENCY ESTIMATION

The problem at hand can be reduced to that of estimating the frequency f_d in the following model[2]

$$\begin{aligned} x(t) &= A.s(t) + w(t) \\ &= Ae^{-2\alpha^2 f_d^2 t^2} e^{j2\pi f_d t} + w(t), \quad t = -N, \dots, N(1) \end{aligned}$$

where $\{w(t)\}$ is assumed to be a complex, circularly symmetric white Gaussian noise. The parameter f_d is directly related to the particle's speed via the relation $f_d = \frac{V}{I}$ where V denotes particle's velocity and I is the interfringe width. The signal $A.e^{-2\alpha^2 f_d^2 t^2} e^{j\omega_d t}$ is of finite energy, preventing from defining the correlation as $\gamma_x(m) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N E \{x^*(k).x(k+m)\}$ (as would be the case for quasi-stationary processes). Here, we define

$$r_x(m) \stackrel{\text{def}}{=} \sum_{k=-N}^{N-m} E \{x^*(k).x(k+m)\} \quad (2)$$

It can be verified that

$$r_s(m) = |A|^2 \gamma(m).e^{-2\alpha^2 f_d^2 m^2} e^{jm\omega_d} + \sigma_w^2 (2N-m+1)\delta_{m,0} \quad (3)$$

where $\gamma(m) \stackrel{\text{def}}{=} \sum_{k=-N}^{N-m} e^{-4\alpha^2 f_d^2 k(k+m)}$ is real-valued. As the phase of (3) is $m\omega_d$, a natural estimate of the frequency ω_d is given by

$$\hat{\omega} = \arg \min_{\omega} \sum_{m=1}^M [\hat{r}_x(m) - m\omega]^2 = C. \sum_{m=1}^M m\hat{\phi}(m) \quad (4)$$

where $C = \frac{6}{M(M+1)(2M+1)}$, $\hat{\phi}(m) = \angle \hat{r}_x(m)$ (herein $\angle y$ denotes the phase of y) and

$$\hat{r}_x(m) = \sum_{k=-N}^{N-m} x^*(k).x(k+m) \quad (5)$$

is an unbiased estimate of $r_x(m)$. $\hat{\omega}$ is obtained by linear regression of the phase $\hat{\phi}(m)$. In (4), M is a user's variable whose choice will be discussed later.

3. ANALYSIS

In this section, we derive the theoretical variance of the estimate (4) under the assumption of small errors in $\hat{\phi}(m)$, that is we consider small deviations from what is called the *equilibrium state*[5]. The equilibrium state is related to the state of model parameters in the noise free case. It should be observed that the high SNR assumption is not actually required in the present analysis. The equilibrium state is defined

by $\phi(m) = \angle r_x(m)$. Let $\tilde{r}_x(m) = \hat{r}_x(m) - r_x(m)$, $\tilde{\phi}(m) = \hat{\phi}(m) - \phi(m)$ and $\tilde{\omega} = \hat{\omega} - \omega_d$ denote the estimation errors, assumed to be small in the sequel. With $C = \frac{6}{M(M+1)(2M+1)}$, the mean of the frequency estimate is readily obtained as

$$\begin{aligned} E\{\hat{\omega}\} &= C. \sum_{m=1}^M m.E\{\hat{\phi}(m)\} = C. \sum_{m=1}^M m.\angle r_x(m) \\ &= C. \sum_{m=1}^M m\angle r_s(m) = C. \sum_{m=1}^M m^2\omega_d = \omega_d \quad (6) \end{aligned}$$

where we used the fact that $\gamma(m)$ is real-valued and $\angle r_x(m) = \angle r_s(m)$ for $m > 0$. We now turn to the variance of $\hat{\omega}$. One has

$$\begin{aligned} \tilde{\phi}(m) &= \hat{\phi}(m) - \phi(m) \\ &= \Im \left\{ \log \left[\frac{\hat{r}_x(m)}{r_x(m)} \right] \right\} \\ &= \Im \left\{ \log \left[1 + \frac{\tilde{r}_x(m)}{r_x(m)} \right] \right\} \\ &\simeq \Im \left\{ \frac{\tilde{r}_x(m)}{r_x(m)} \right\} \quad (7) \end{aligned}$$

where the symbol $\simeq$ is used to denote an approximation in which the neglected terms tend to zero at a faster rate than the retained ones. With $\lambda(k, m) = \frac{1}{A} e^{2\alpha^2 f_d^2 (m^2 - k^2)} e^{-j(k+m)\omega_d}$,

$\mu(k, m) = \frac{1}{A^*} e^{-2\alpha^2 f_d^2 k(2m+k)} e^{jk\omega_d}$ and $\nu(m) = \frac{1}{|A|^2} e^{2\alpha^2 f_d^2 m^2} e^{-jm\omega_d}$, it can be shown that

$$\begin{aligned} \tilde{\phi}(m) &= \frac{1}{\gamma(m)} \sum_{k=-N}^{N-m} \Im \{ \lambda(k, m).w(k+m) \\ &\quad + \mu(k, m).w^*(k) + \nu(m).w^*(k).w(k+m) \} \\ &:= \frac{1}{\gamma(m)} \sum_{k=-N}^{N-m} \Im \{ f(k, m) \} \quad (8) \end{aligned}$$

Therefore,

$$\begin{aligned} \text{var}\{\hat{\omega}\} &= C^2. \sum_{m,n=1}^M mn.E\{\tilde{\phi}(m).\tilde{\phi}(n)\} \\ &= C^2 \sum_{m,n=1}^M \frac{mn}{\gamma(m).\gamma(n)} \times \\ &\quad \sum_{k=-N}^{N-m} \sum_{p=-N}^{N-n} E\{\Im[f(k, m)] \Im[f(p, n)]\} \quad (9) \end{aligned}$$

After some calculations, the theoretical variance of the frequency estimate can be obtained as

$$\text{var}(\hat{\omega}) = T_1 + T_2 + T_3 + T_4 \quad (10)$$

where

$$T_1 = -\frac{C^2 \cdot \sigma_w^2}{|A|^2} \sum_{m,n=1}^M \frac{mn \cdot \gamma(m+n)}{\gamma(m) \cdot \gamma(n)} e^{-4\alpha^2 f_d^2 \cdot nm} \quad (11)$$

$$T_2 = \frac{C^2 \cdot \sigma_w^2}{2|A|^2} \sum_{m,n=1}^M \frac{mn}{\gamma(m) \cdot \gamma(n)} \times \sum_{k=\max(-N, -N+m-n)}^{N-m} e^{-4\alpha^2 f_d^2 \cdot (k-n)(k+m)} \quad (12)$$

$$T_3 = \frac{C^2 \cdot \sigma_w^2}{2|A|^2} \sum_{m,n=1}^M \frac{mn}{\gamma(m) \cdot \gamma(n)} \times \sum_{k=-N}^{\min(N-n, N-m)} e^{-4\alpha^2 f_d^2 \cdot k \cdot (k+n+m)} \quad (13)$$

$$T_4 = \frac{C^2 \sigma_w^4}{2|A|^4} \sum_{m=1}^M \frac{(2N+1-m)m^2}{\gamma^2(m)} e^{4\alpha^2 f_d^2 m^2} \quad (14)$$

Expressions (11)-(14) enable to compute the theoretical variance of $\hat{\omega}$ for any given value of N, M, SNR, α, f_d . It is to be observed that, although the phase of the correlation does not actually depend on the time-varying amplitude, this latter has obviously an influence on the variance of the frequency estimate. Additionally, we note that the variance is the sum of four terms: the three first ($T_1 + T_2 + T_3$) depend on $SNR^{-1} = \sigma_w^2 / |A|^2$ whereas T_4 is proportional to SNR^{-2} .

4. NUMERICAL EXAMPLES

4.1. Simulated data

The aim of this section is twofold. Firstly, we check the validity of the theoretical analysis by comparing the theoretical variance, as given by (10), to empirical variances obtained through Monte-Carlo simulations. Secondly, we compare the variance of the frequency estimator to the Cramér-Rao Bounds. Additionally, the simulations presented here illustrate the influence of various parameters onto the estimator performance. They enable to derive "optimal" values for N and M . Unless otherwise specified, the value of α is selected as $\alpha = 0.122857$ and $A = e^{j\varphi}$ (where φ is uniformly distributed on $[0, 2\pi[$) thorough the simulations. The Signal to Noise Ratio (SNR) is defined as

$SNR = \frac{|A|^2}{\sigma_w^2}$. For each simulation, 500 Monte-Carlo trials are run to estimate the variance of the estimator. We begin with studying the influence of N . Figure 1 compares the theoretical and empirical variances of the estimator with the CRB for varying N .

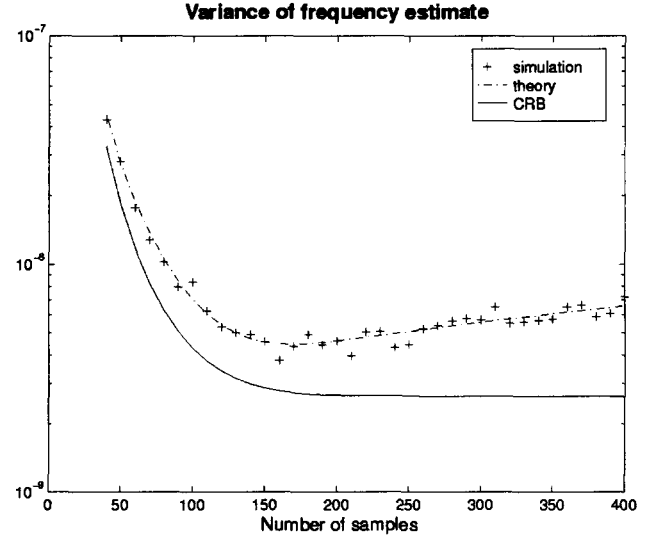


Figure 1: CRB, theoretical and empirical variances of the frequency estimator versus N . $f_d = 0.05$. $SNR = 10dB$. $M = 15$.

First, it can be seen that the theoretical and empirical results are in very good agreement. Additionally, the frequency estimator proposed herein is seen to come close to the Cramér-Rao bound, if N is not too large. It should be noted that, for N above a threshold (typically $N > 1.5/\alpha f_d$), the frequency estimator proposed here shows an increase in the error variance. Observe that a good rule of thumb is to process as many samples as are recorded during the crossing of a particle through the entire probe volume. Next, we study in Figure 2 the influence of the parameter M , the number of correlation lags used in the linear regression.

We note that the variance of the frequency estimate decreases with increasing M and reaches the CRB. Moreover, for M above a threshold (*i.e.* $M > N/3$) the variance does not decrease. The previous figures suggest that there exists some "optimal" choice of the couple (N, M) for which the errors achieve their lower bound, which is shown to be the CRB. Many other numerical simulations (not reported here) lend support to the fact that an "optimal" choice is given by

$$\begin{aligned} N_{opt} &\simeq \frac{3}{2\alpha f_d} = \frac{3}{2} \frac{W}{V} \\ M_{opt} &\simeq \frac{N_{opt}}{3} = \frac{1}{2\alpha f_d} = \frac{W}{2V} \end{aligned} \quad (15)$$

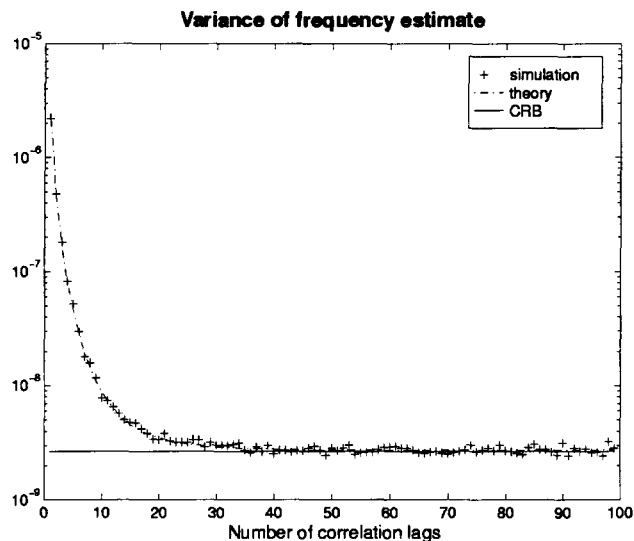


Figure 2: CRB, theoretical and empirical variances of the frequency estimator versus M . $f_d = 0.05$. $SNR = 10dB$. $N = 200$.

The choice in (15) provides a simple way of selecting both the time window and the number of correlation lags used in the regression algorithm. With this choice, the estimator proposed herein attains the CRB over a wide range of scenarios. Figure 3 illustrates this claim by plotting the variance of the proposed estimator as a function of f_d when N and M are chosen as in (15).

4.2. Application to real data

We now illustrate the effectiveness of the estimator proposed here on real data recorded in a wind tunnel. Particles of air were injected with a known velocity. The signal received by the photodetector was recorded and stored for post-treatment. The estimator was applied to the data with the selection rule for N and M given by (15). Four cases are presented here, which correspond to different particles's speeds ranging from $100ms^{-1}$ (case #1) to $65ms^{-1}$ (case #4). For comparison purposes, the MLE was applied to the same set of data. From the MLE, consistent estimates of the CRB were computed in order to check if the frequency estimate proposed here belongs to the interval $[f_{MLE} - \sqrt{CRB}, f_{MLE} + \sqrt{CRB}]$. Table 1 lists the values of the frequency estimates obtained by the phase-based method and MLE.

Case#	1	2	3	4
f_{MLE}	$4.127348E-2$	$4.01222E-2$	$3.245991E-2$	$2.773188E-2$
f_{PHASE}	$4.12981E-2$	$4.012169E-2$	$3.246473E-2$	$2.771189E-2$
$\frac{f_{PHASE} - f_{MLE}}{\sqrt{CRB}}$	0.703	-0.018	0.372	-0.855

Table 1: Values of the frequency estimate obtained from the phase-estimator and MLE

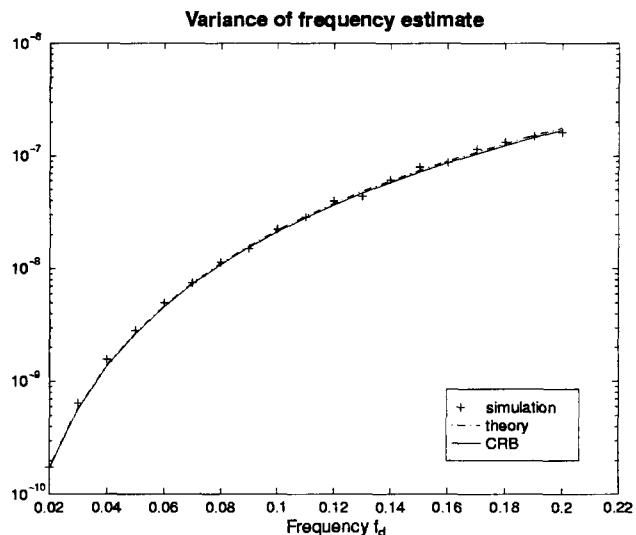


Figure 3: CRB, theoretical and empirical variances of the frequency estimator versus f_d . $N = \frac{3}{2\alpha f_d}$. $M = \frac{N}{3}$. $SNR = 10dB$.

As can be seen, the estimator proposed here provides a frequency close to that obtained by the MLE, i.e. within the $\pm\sqrt{CRB}$ interval.

5. REFERENCES

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