Review

- Matched filter h_{-k}^{*}

  - implements a minimum-distance receiver which is the optimal criterion for additive white Gaussian noise

- Reflected Transfer Function $H^*(1/z^*)$
- Folded Spectrum $$S_h(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty}\left|H(f-\frac{m}{T})\right|^2$$
- Minimum-phase spectral factorization:

$$S(z) = \gamma^2 M(z) M^*(1/z^*)$$

γ^2 … geometric mean of $S(e^{j\Theta})$

S(z) … rational, real/nonnegativ on the unit circle

M(z) … monic, loosely minimum-phase

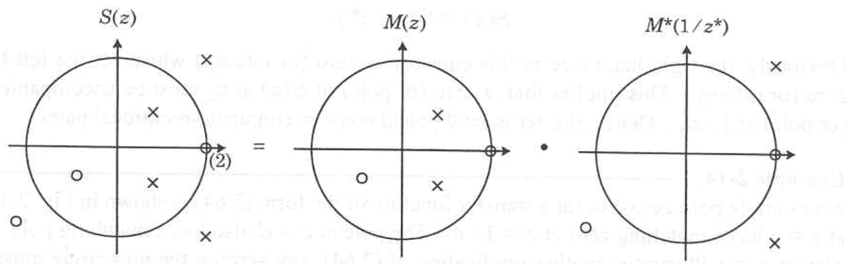
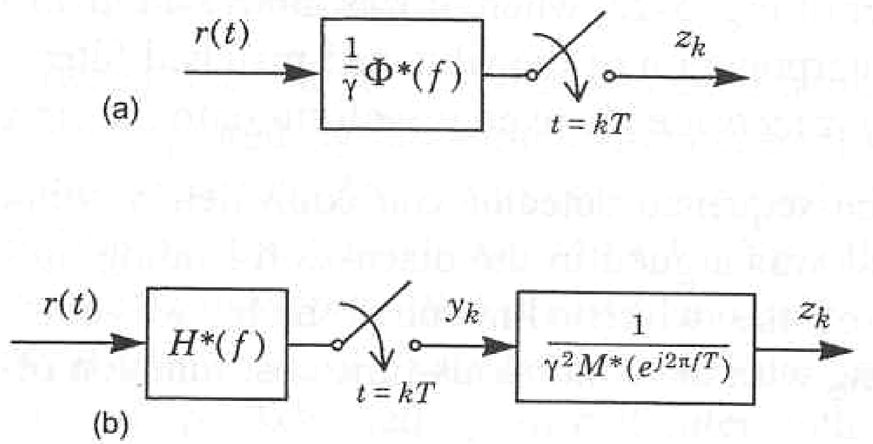
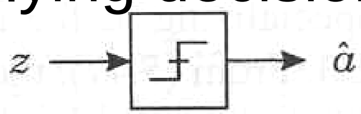


Fig. 2-13. Spectral factorization of a transfer function $S(z)$, which is non-negative real on the unit circle. The zero at $z = 1$ has multiplicity two (in general its multiplicity could be any even integer). These two zeros on the unit circle are split between $M(z)$ and $M^*(1/z^*)$.

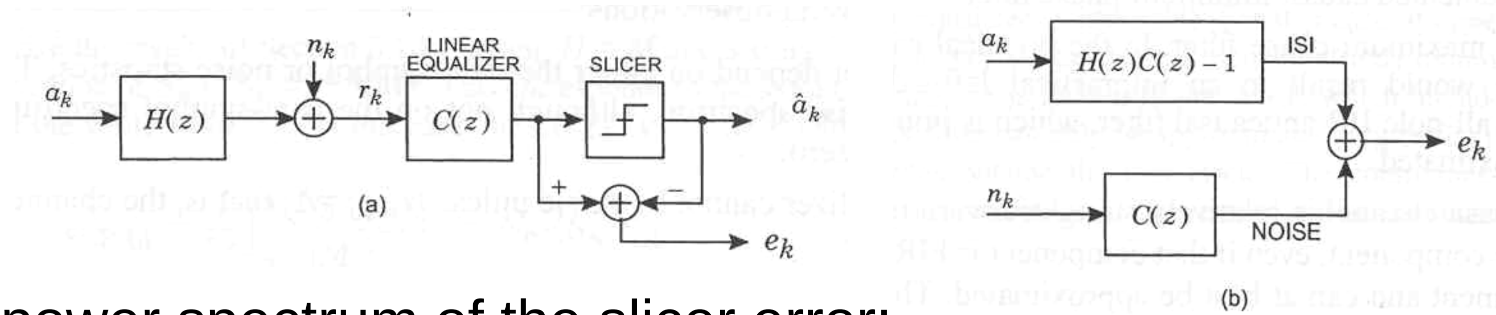
Whitened Matched Filter (WMF), Slicer



- **WMF** provides sufficient statistics for the minimum distance receiver
- transforms white noise into white noise
  i.e. noise power spectrum after MF: $N_0 S_h(z)$
  after precursor equalizer: N_0 / γ^2
- output is causal and monic and minimum-phase
- **Slicer:** quantizes the input to the nearest alphabet symbol (e.g. by applying decision thresholds)



Linear Equalizer (LE)



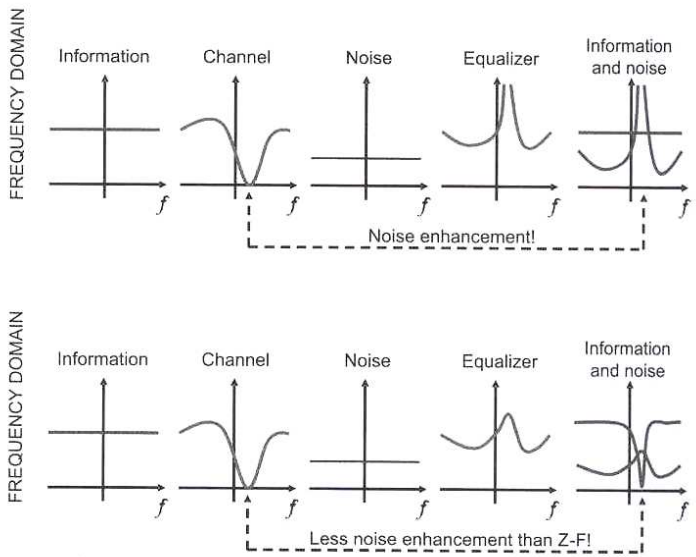
- power spectrum of the slicer error:

$$S_e = S_a |HC - 1|^2 + S_n |C|^2$$

- MSE: $E\left[|e_k|^2\right] = \varepsilon^2 = \langle S_e \rangle_A$

a)

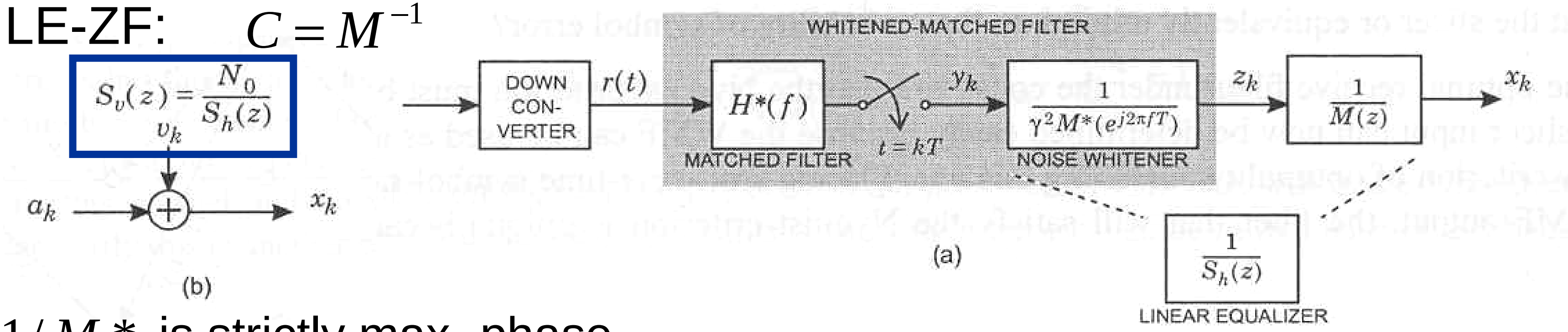
b)



- **Zero – forcing (ZF) Criterion (a):**
  - forces the ISI component of the slicer error to zero
- **Mean Square Error (MSE) Criterion (b):**
  - minimize the MSE – i.e. ISI and noise together

LE-ZF with WMF front end

- White Gaussian noise assumed
- LE-ZF: $C = M^{-1}$



- $1/M^*$ is strictly max.-phase

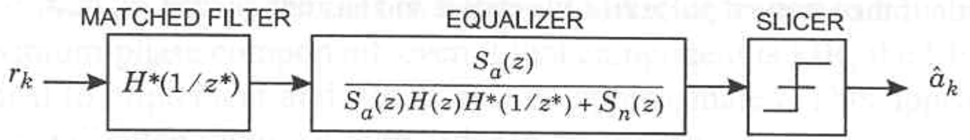
  i.e. if M has zeros -> WMF has poles outside the unit circle
- Continuous-time MF problematic if h(t) is causal with unbounded support
- for a general channel model:

$$C = \frac{1}{H} = \frac{1}{H_0 H_{\min} H_{\max} H_{zero}} \qquad \varepsilon^2_{ZF-LE} = \left\langle S_n / |H|^2 \right\rangle_A$$

- problems with non-minimum-phase channels (noise at slicer input)
- problems when zeros/poles approach the unit circle

LE-MSE

- power spectrum of the channel output: $S_r = S_a |H|^2 + S_n$
- task: minimize $S_e = S_r \left| C - S_a S_r^{-1} H^* \right|^2 + S_a S_n S_r^{-1}$
- leads to: $C = S_a S_r^{-1} H^*$



- assume a channel with poles => anticausal IIR matched filter
- MSE:

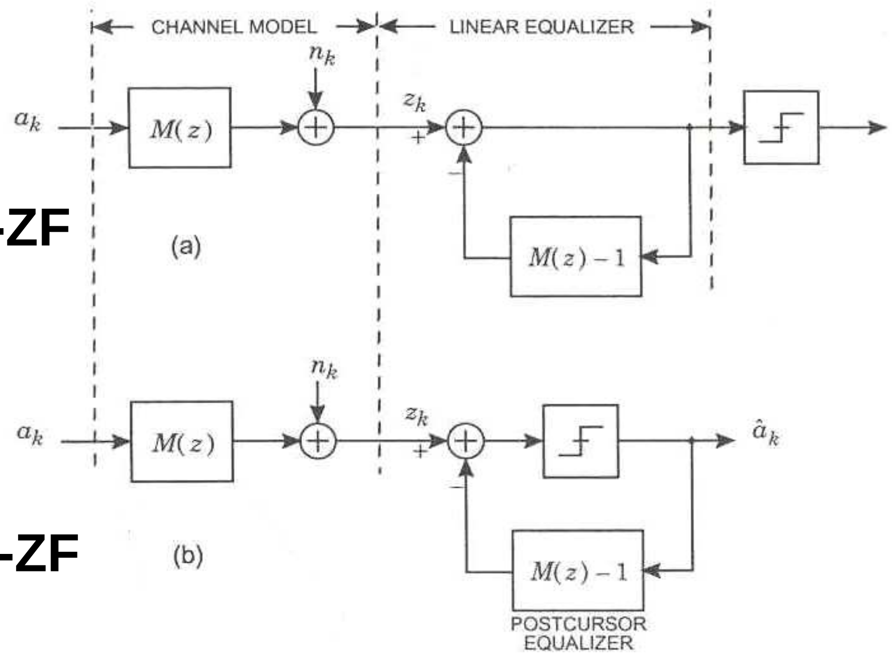
$$\varepsilon^2_{MMSE-LE} = \left\langle \underbrace{S_n / (|H|^2}_{S_{e,LE-ZF}} + S_n S_a^{-1}) \right\rangle_A$$

- for $S_n \to 0$ the LE-MSE approaches the LE-ZF
- problems with channel-poles (except at $z = 0 / z = \infty$)

Decision Feedback Equalizer (DFE)

LE-ZF

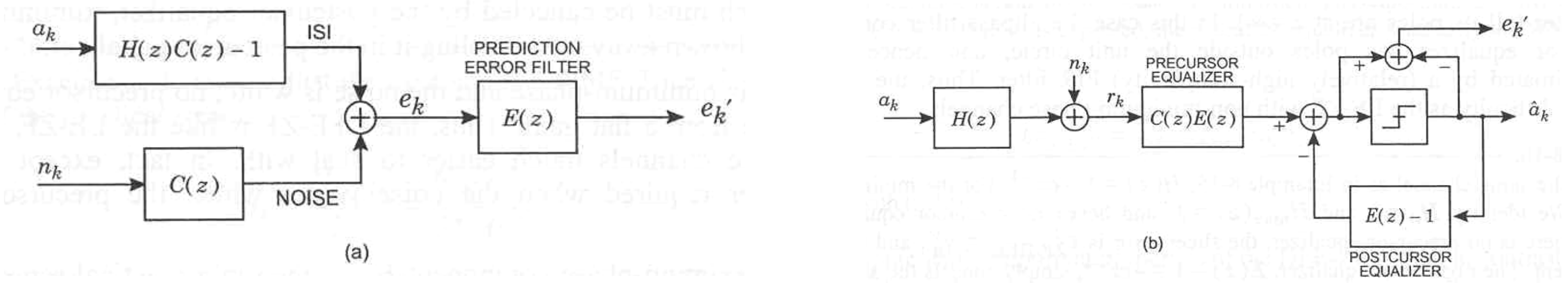
DFE-ZF



- postcursor equalizer eliminates ISI introduced by past samples
- slicer removes noise (-> noise reduction) and has bounded output (-> stability!)
- risk of error propagation

DFE - ZF

- correlated slicer error samples -> **linear prediction error filter** E(z) (causal, monic) reduces noise variance and results in a white DFE slicer-error



- LE slicer error: $S_e = \varepsilon_{DFE}^2 \cdot M_e M_e^*$ where $\varepsilon_{DFE}^2 = \langle S_e \rangle_G$

$$= \frac{S_n}{HH^*} = \frac{\gamma_n M_n M_n^*}{|H_0^2| \cdot H_{\min} H_{\max} H_{\min}^* H_{\max}^*}$$

$$E = \frac{H_{\min} H_{\max}^*}{M_n}$$

precursor equalizer: $CE = \frac{1}{H_0} \underbrace{\frac{H_{\max}^*}{H_{\max}}}_{allpass\_filter} M_n^{-1}$

$$C = H^{-1}$$

$$\varepsilon_{ZF-DFE}^2 = \left\langle S_n / |H|^2 \right\rangle_G$$

DFE-ZF (cont‘d)

- DFE-ZF relies on a minimum-phase equivalent channel (this minimizes the noise at the slicer input) -> WMF
  - because:

    - the DFE‘s decision relies on the first sample of the impulse and ignores the signal energy embedded in the ISI terms

    - the MLSD uses all the energy in the equivalent channel impulse response
  - therefore: DFE relies on the spectral factorization!
- decision rule:

$$\hat{x}_k = \arg\min_{x \in X} \left| y_k - \sum_{i=1}^{\eta} m_i \hat{x}_{k-i} - x \right|^2$$

- correlates to a VA working on a trellis with only one state

DFE - MSE

- C is again chosen to minimize S_e as in the LE-MSE
- channel output: $S_r = S_a HH^* + S_n = \gamma_r^2 \cdot M_r M_r^*$
- error before LP: $\hat{S}_e = S_r \left| C - S_a S_r^{-1} H^* \right|^2 + S_a S_n S_r^{-1}$
  - -> same C as for the LE-MSE: $C = S_a S_r^{-1} H^*$
- again we look for a filter E that whitens the slicer error:

$$S_e = \frac{S_a S_n}{S_r} = \frac{\gamma_a \gamma_n}{\gamma_r} \cdot \frac{M_a M_n M_a^* M_n^*}{M_r M_r^*} = \varepsilon^2_{MMSE-DFE} \cdot M_e M_e^*$$

$$E = \frac{1}{M_e} = \frac{M_r}{M_a M_n}$$

- optimal precursor equalizer: $CE = \frac{\gamma_a^2}{\gamma_r^2} \cdot H^* \cdot \frac{M_a^*}{M_r^*} \cdot M_n^{-1}$
  - it includes: matched filter (cp. LE-MSE), noise-whitening filter (cp. DFE-ZF)
- MSE: $\varepsilon^2_{MMSE-DFE} = \left\langle S_n / (|H|^2 + S_n S_a^{-1}) \right\rangle_G$

MLSE

- Maximum likelihood sequence estimation is the optimal minimum probability of error detector on ISI channels
- Maximum likelihood (i.e. Minimum distance) rule:

$$\hat{x}(z) = \underset{x(z)\in \mathrm{X}[z]}{\arg\min} \left\| y(z) - M(z)x(z) \right\|^2$$
$$= \underset{x(z)\in \mathrm{X}[z]}{\arg\min} \left\| y(z) - \underbrace{(M(z)-1)x(z)}_{ISI} - x(z) \right\|^2$$

- where $\mathrm{X}[z] = \{x_0 + x_1 z + \ldots + x_{N-1} z^{N-1} \mid x_k \in \mathrm{X}\}$

- number of states is exponential with alphabet size and channel length K, i.e. $|\mathrm{X}|^K$
- Viterbi Algorithm searches a state sequence through the trellis that minimizes this distance

from MLSE to RSSE

- in MLSE: state defined as $p_n = [x_{n-1}, x_{n-2}, \ldots, x_{n-K}]$
- RSSE … „Reduced State Sequence Estimation“
  - each subset-state in RSSE consists of the union of serveral ML states
  - for x_{n-k} define a 2-dim. set partitioning $\Omega(k)$
    where the signal set is partitioned into J_k subsets $(1 \leq J_k \leq M)$
  - conditions: $J_1 \geq J_2 \geq \ldots \geq J_K$
    $\Omega(k)$ is a further partition of the subsets of $\Omega(k+1)$
  - $a_i(k)$ … index of the subset of a symbol
  - subset state of a sequence at time n:
    $t_n = [a_{n-1}(1), a_{n-2}(2), \ldots, a_{n-K}(K)]$
  - $\prod_{k=1}^{K} J_k$ states in the subset trellis
  - J_1 transitions per state (parallel transitions when $J_1 < M$)

RSSE (cont‘d)

- certain paths will merge earlier than in the ML trellis

  -> set partitioning should be such that these paths can be reliably distinguished at the point of merging

- for every $\Omega(k)$, maximize the min. intrasubset Euclidean distance

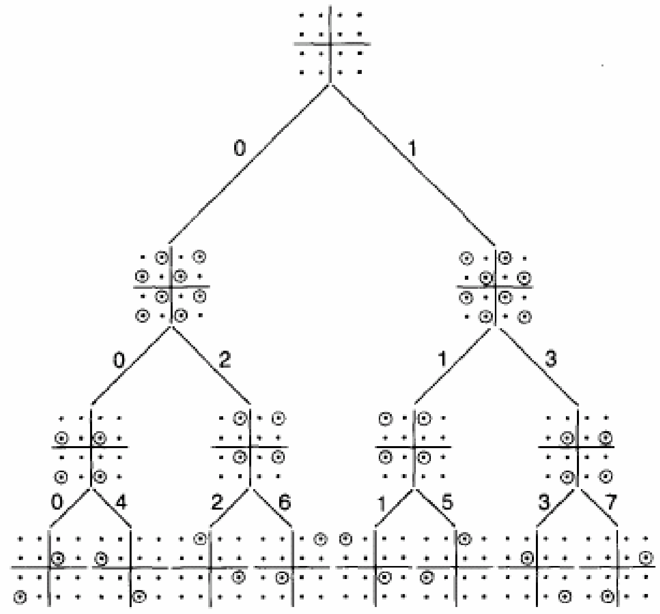
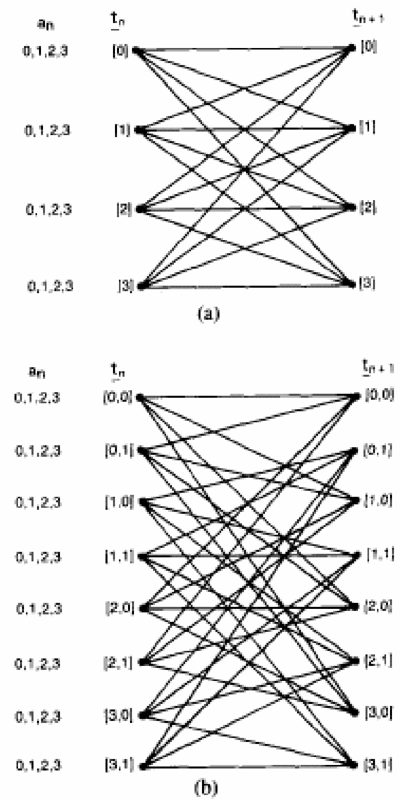


Fig. 2. Ungerboeck partition tree for the rectangular 16-QAM signal set.



Subset trellises (a) $K = 1$, $J_1 = 4$, (b) $K = 2$, $J_1 = 4$, $J_2 = 2$.

RSSE (cont‘d)

- when $J_1 < M$, then for each subset transition the VA can first select the symbols with the minimum branch metric
- branch metrics: $\left| y_n - (\hat{p}_n(t_n), f) - x_n \right|^2$
- $\hat{p}_n(t_n)$ represents the K most recent symbols stored in the path history associated with the state t_n
- in contrast to MLSE, performance of RSSE is affected by phase response

  for $J_k = 1$ for all k, RSSE degenerates into ZF-DFE
- special case:

$$J_k = \begin{cases} M & 1 \le k \le K' \\ 1 & K' \le k \le K \end{cases}$$

… „delayed decision feedback sequence estimator“ (DDFSE)

DDFSE

- hybrid of MLSE and ZF-DFE (with each as special cases)
- VA on a **truncated channel impulse response** and using ZF-DFE on each branch of the trellis to remove ISI
- branch metric calculation:

$$\left| y_k - \underbrace{\underbrace{\sum_{i=1}^{\mu} m_i x_{k-i}}_{state-contribution} - \underbrace{\sum_{i=\mu+1}^{\eta} m_i \hat{x}_{k-i}}_{delayed-decision-contribution}}_{ISI} - x_k \right|^2$$

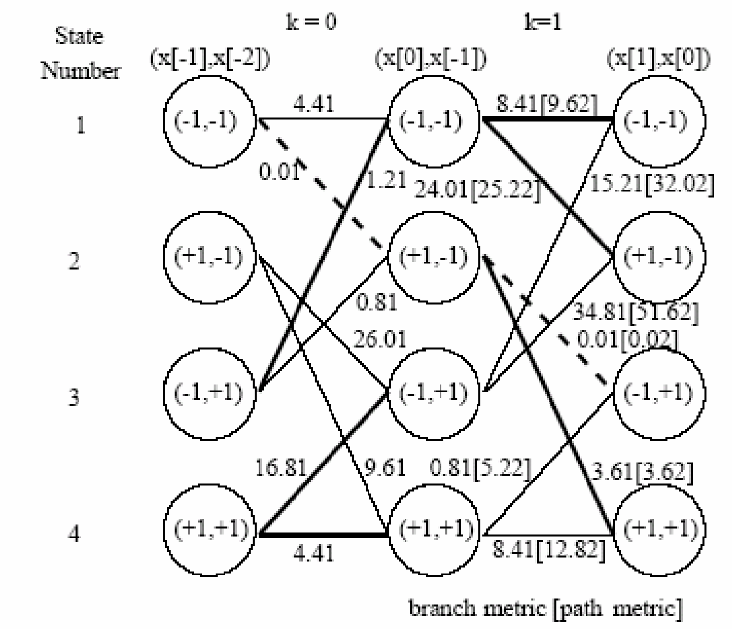
- channel transfer function: $M(z) = M_\mu(z) + z^{\mu+1} M^+(z) = \dfrac{\beta(z)}{\gamma(z)}$
- further: $M^+ = \dfrac{\beta^+(z)}{\gamma(z)}$; $n = \deg\{\beta^+(z)\}$; $m = \deg\{\gamma(z)\}$

DDFSE (cont'd)

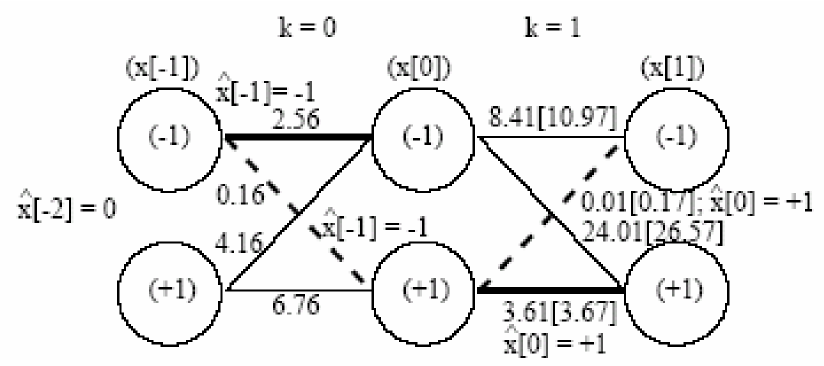
- branch metric: $$\left| y_k - \sum_{i=0}^{\mu} m_i x_{k-i} - \hat{w}_{k-\mu-1} \right|^2$$
- where $$\hat{w}_{k-\mu-1} = \begin{cases} \sum_{i=0}^{n} \beta_i^{+} \hat{x}_{k-\mu-1-i} - \sum_{i=1}^{m} \gamma_i \hat{w}_{k-\mu-1-i} & (m > 0) \\ \sum_{i=0}^{\eta-\mu-1} m_i \hat{x}_{k-\mu-1-i} & (m = 0) \end{cases}$$
- therefore each branch metric calculation requires:
  - current state $X_k = (x_{k-1}, x_{k-2}, \ldots, x_{k-\mu})$
  - previous n decisions $\{\hat{x}_{k-\mu-1}, \ldots, \hat{x}_{k-\mu-n}\}$
  - previous m estimates $\{\hat{w}_{k-\mu-2}, \ldots, \hat{w}_{k-\mu-m-1}\}$

DDFSE (cont‘d)

- example:
  - FIR channel: $M(z) = 1 - 1.5z^{-1} + 0.5z^{-2}$
  - received sequence: $Y(z) = 2.1 - 2.9z^{-1}$
  - state of the DDFSE trellis (μ=1): $\mathrm{X}_k = x_{k-1}$
  - decision feedback contribution: $\hat{w}_{k-2} = 0.5\hat{x}_{k-2}$
- branch and path metrics for the **VA MLSE**:



- and for the **DDFSE**:



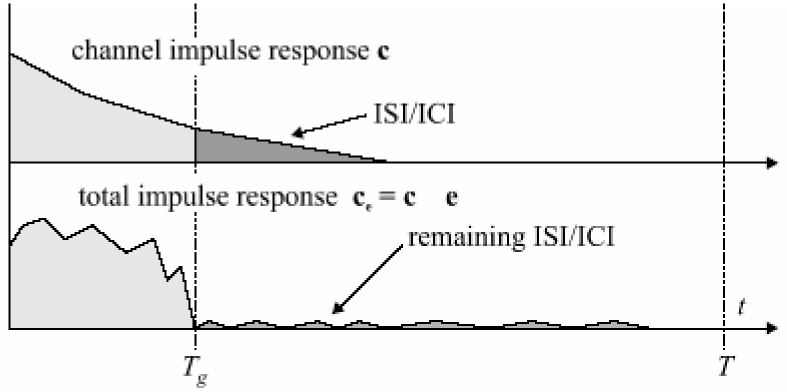
impulse response truncation

- m … length of impulse response c
- $e = [e(0), e(1), \ldots, e(p)]^T$ … FIR pre-equalizer with length p
- convolution yields: $c_e = c * e = [c_e(0), c_e(1), \ldots, c_e(m+p-1)]^T = F \cdot e$
- task: minimize the power of all $c_e(k)$ for $k > N_g$
- $c_e(k) \overset{!}{=} \begin{cases} 1 & k = 0 \\ * & 0 < k \le N_g \\ 0 & k > N_g \end{cases}$

$$F_r \cdot e = d + \delta$$

$$d = [1,0,0,\ldots,0]^T$$

- $F_r = F([1, N_g + 1 : m + p - 1], :)$
  … reduced convolution matrix
- d … destination vector
- $\delta = [\delta(0), \delta(1), \ldots, \delta(m + p - N_g)]$
  … error vector



impulse response truncation (cont'd)

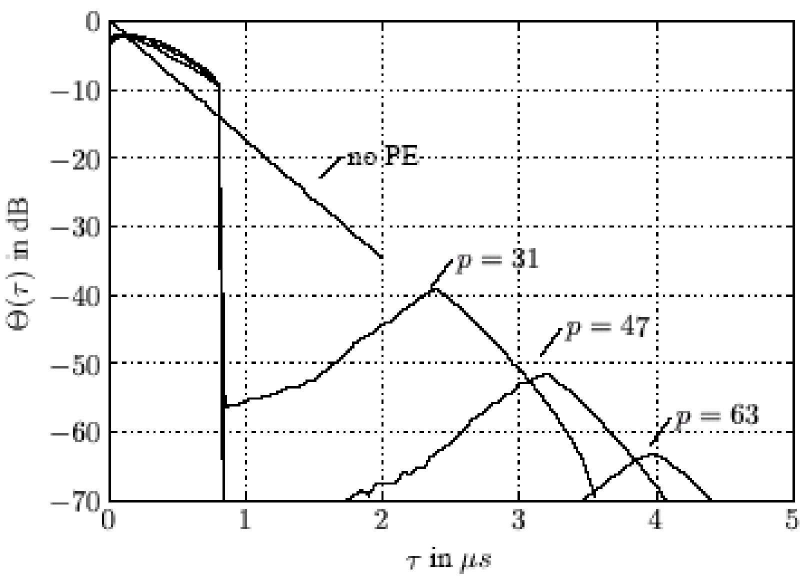
- MMSE technique leads to

$$e = (F_r \cdot F_r^T)^{-1} \cdot F_r^T \cdot d \qquad ; \qquad \gamma_{pe} = 1/SNR$$

- under AWGN:

$$e = (F_r \cdot F_r^T + \gamma_{pe}^2 \cdot I)^{-1} \cdot F_r^T \cdot d$$

- power delay spectrum:
  - no noise, ideal channel knowledge
  - based on more than 5000 random channels
  - exponential power delay profile
  - power delay spread $\Delta\tau = 250ns$



Allpass prefilter computation

- closed form computation necessary (short estimation training sequences)
- $$H_{\min}(z) \cdot H_{\min}^{*}(1/z^{*}) = H(z) \cdot H^{*}(1/z^{*})$$
- $$A(z) = \frac{H_{\min}(z)}{H(z)}$$
- one possibility: calculating $H_{\min}(z)$ (spectral factorization, root finding, prediction-error filter,…)
- A(z) is non-stable
  - -> noncausal, stable -> truncation -> causal FIR with delay
  - time-reversal -> A(1/z) -> time-reversal of the output
  - $\tilde{A} = H_{\max}(z) / H(z)$ -> reduced-state equalization in negative time direction (backward decoding)

Allpass prefilter computation (cont‘d)

- another possibility: apply the **FIR feedforward filter of a MMSE-DFE** (not robust to a mismatch of design parameters in certain cases – virtual noise variance, delay, filter length etc.)
- prefilter computation **based on Linear Prediction**:
- $\frac{H_{\min}(z)}{H(z)} = \frac{H^*(1/z^*)}{H^*_{\min}(1/z^*)}$ $\quad A(z) = A_1(z) \cdot A_2(z) = H^*(1/z^*) \cdot \frac{1}{H^*_{\min}(1/z^*)}$
- approx. by FIR filter (noise whitening filter):

$$F_2(z) \approx C \frac{1}{H^*_{\min}(1/z^*)}$$

- equivalently: $F_2(z) = G^*(1/z^*)$, $G(z) \approx C^* \frac{1}{H_{\min}(z)}$
- choice: $G(z) = 1 - P(z)$ … prediction-error filter of order q_p

Allpass prefilter computation (cont‘d)

- optimum coefficients minimize the output power of the error filter
- -> solution of the Yule-Walker equations: $\Phi_{hh} p = \varphi_{hh}$
  - Φ_{hh} … correlation matrix
  - φ_{hh} … correlation vector
  - p … coefficients of the prediction filter
- can be recursively solved using the Levinson-Durbin Algorithm
- $1 - P(z)$ and $1/H_{\min}(z)$ are causal and min. phase
- for infinite filter order ($q_p \to \infty$): $G(z) = 1 - P(z) = \dfrac{C^*}{H_{\min}(z)}$
- -> overall transfer function of the FIR prefilter:

$$F(z) = z^{-(q_h + q_p)} \cdot H^*(1/z^*) \cdot (1 - P^*(1/z^*))$$