Maximally Decimated Filterbanks

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Some Terms

- **Subband Coding:** Dividing and reconstructing a signal in/from multiple subbands.
- **Maximally Decimated Filter Bank:** The samplerates of the subbands of a filter bank will be reduced to the Minimum ($2 \cdot$ Nyquist-Frequency)
- **Quadrature Mirror Filter (QMF):** Early form of a 2-Channel Maximally Decimated Filter Bank - aliasing is internally permitted but cancelled at the output. Today also used very generally for M-channel filter banks containing this concept.

QMF-Structure

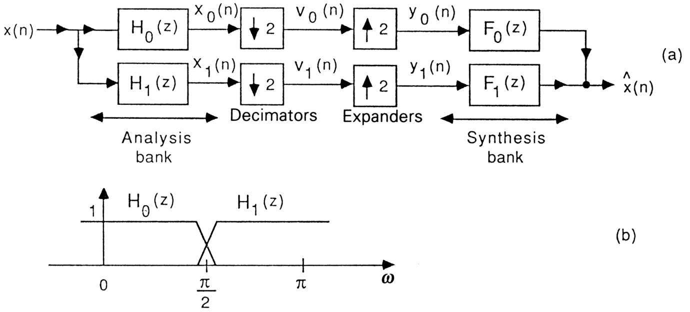


Figure 5.1-1 (a) The quadrature mirror filter bank and (b) typical magnitude responses.

Errors Created In QMF-Banks 1

- **Amplitude Distortions**
- **Phase Distortions**
- **Aliasing**
- Coding and Quantization Artifacts

Errors Created In QMF-Banks 2

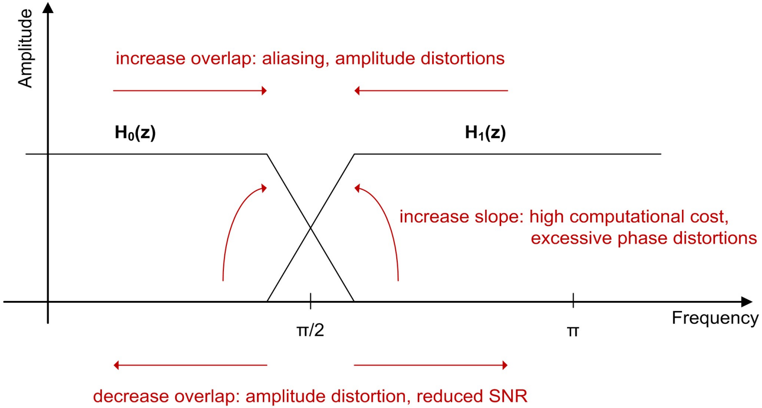
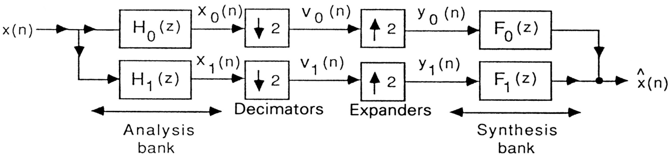


Figure: Errors in Filter Banks

Transfer Function 1



$$
\begin{aligned}
X_k(z) &= H_k(z)X(z) \qquad , k = 0, 1 \\
V_k(z) &= \frac{1}{2}[X_k(z^{\frac{1}{2}}) + X_k(-z^{\frac{1}{2}})] \\
Y_k(z) &= V_k(z^2) = \frac{1}{2}[H_k(z)X(z) + H_k(-z)X_k(-z)] \\
\hat{X}(z) &= F_0(z)Y_0(z) + F_1(z)Y_1(z)
\end{aligned}
$$

Transfer Function 2

$$\begin{aligned}
\hat{X}(z) &= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) \\
&\quad + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z) \\
A(z) &= \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)] \rightarrow 0 \qquad \textit{Aliasing TF} \\
T(z) &= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)] \qquad \textit{Distortion TF}
\end{aligned}$$

Aliasing Cancellation

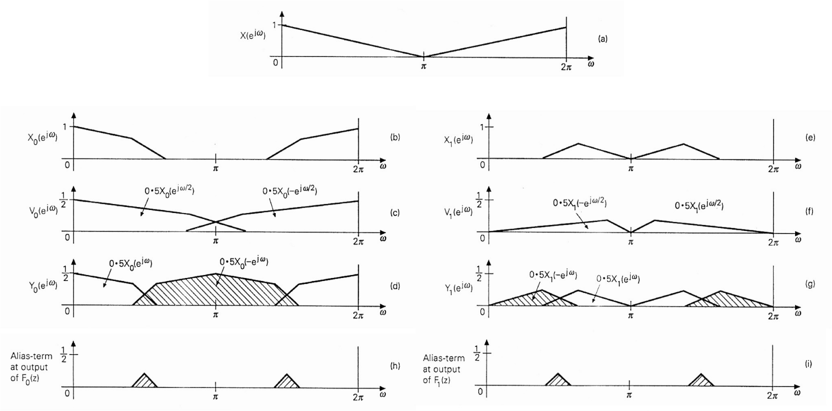


Figure 5.1-3 Various internal signals, and alias cancelation mechanism in the QMF bank. (© Adopted from 1990 IEEE.)

Distortion-Free Conditions

if aliasing is cancelled:

$$\hat{X}(e^{j\omega}) = |T(e^{j\omega})| e^{j\phi\omega} X(e^{j\omega})$$

- **Free from Amplitude Distortion:**
  $|T(e^{j\omega})| = d, \quad d \neq 0 \text{ } \textit{for all } \omega$
- **Free from Phase Distortion:**
  $\phi(\omega) = a + b\omega, \quad a, b = const$
  $\Rightarrow$ only linear phase components
- **Perfect Reconstruction (PR):**
  $T(z) = cz^{-m_0}, \quad c = const$
  $\Rightarrow$ only a time delay

"Classic" QMF: Preferences

$$\hat{X}(z) = \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z)$$

Simple Choice:

$H_1(z) = H_0(-z)$

$F_0(z) = H_0(z), \quad F_1(z) = -H_1(z) = -H_0(-z)$

$$T(z) = \frac{1}{2}[H_0^2(z) - H_0^2(-z)]$$
$$A(z) = 0$$

"Classic" QMF: Eliminating Phase Distortion

$$T(z) \;=\; \frac{1}{2}[H_0^2(z) - H_0^2(-z)]$$

$H_0(z) =$ linear phase $\Rightarrow H_0^2(z) =$ linear phase
$\Rightarrow H_0^2(-z) =$ linear phase
$\Rightarrow T(z) =$ linear phase

- Only possible with FIR-filters
- Just **minimization** of amplitude distortion possible, eg. with costfunction $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$ or other numerical solutions

"Classic" QMF: Eliminating Amplitude Distortion 1

⇒ T(z) is allpass and IIR

$$T(z) = \frac{1}{2}[H_0^2(z) - H_0^2(-z)]$$

not very usable ⇒ polyphase form:

$$\begin{aligned} H_0(z) &= E_0(z^2) + z^{-1}E_1(z^2) \\ T(z) &= \frac{1}{2}[H_0^2(z) - H_0^2(-z)] \\ &\vdots \\ T(z) &= 2z^{-1}E_0(z)E_1(z) \end{aligned}$$

⇒ Convolution of two allpass-filters is also an allpass-filter

⇒ $E_0(z)$ and $E_1(z)$ "just" have to be allpass

"Classic" QMF: Eliminating Amplitude Distortion 2

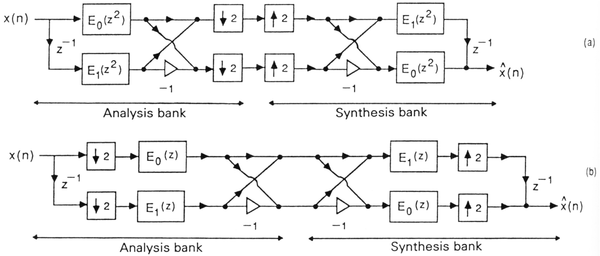


Figure 5.2-2 (a) The complete QMF bank in polyphase form. (b) Rearrangement using noble identities.

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1}E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

"Classic" QMF: Eliminating Amplitude Distortion 3

Necessary for deconstruction into 2 polyphase allpass filters:

- Power symmetric TF

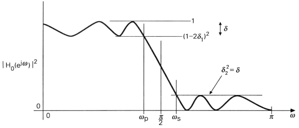


Figure 5.2-6 Square of the magnitude response function for a power symmetric filter.

- Symmetric numerator of TF

⇒ **Butterworth** and **Elliptic Filters** are possible

⇒ steep and very efficient filters, but inherent phase distortion

"Classic" QMF: FIR Perfect Reconstruction

FIR power symmetric filter:

$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}[H_0(z)H_1(-z) + H_1(z)H_0(-z)]X(z) \\ H_1(z) &= z^{-N}\tilde{H}_0(-z), \qquad N \geq \textit{Order of } H_0,\ N \textit{ odd} \\ \tilde{H}_0(z) &= H^{\dagger}(1/z^{*}), \qquad \textit{transpose conjugate}\ldots\end{aligned}$$

$$\hat{\mathbf{X}} = \mathbf{-z^{-N}X(z)}$$

Structure

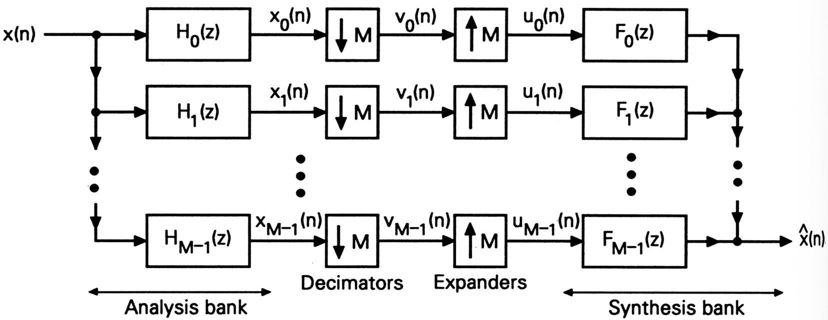


Figure 5.4-1 The M-channel (or M-band) maximally decimated filter bank. Also called M-channel QMF bank.

Basic Transfer Functions 1

$$\hat{X} = \sum_{l=0}^{M-1} A_l(z) X(zW^l), \quad W = e^{-j\frac{2\pi}{M}}$$

$$A_l(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(zW^l) F_k(z)$$

Aliasing is eliminated, if:

$$A_l(z) = 0 \qquad \textit{for } 1 \leq l \leq M-1$$

$$T(z) = A_0(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(z) F_k(z)$$

"Choose synthesis filters F_k such that overlapping terms cancel out" [1]

Basic Transfer Functions 2

Vector Form:

$$M \begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix} = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{M-1}) & H_1(zW^{M-1})) & \cdots & H_{M-1}(zW^{M-1}) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix} =$$

$$\mathbf{t}(z) = \begin{bmatrix} MT(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \left(= \begin{bmatrix} az^{-m_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \textit{ for perfect reconstruction} \right)$$

Basic Transfer Functions 3

Objective:

$$\begin{aligned} \mathbf{H}(z)\cdot\mathbf{f}(z) &= \mathbf{t}(z) \\ \Rightarrow \mathbf{f}(z) &= \mathbf{H}^{-1}(z)\mathbf{t}(z) \\ \mathbf{f}(z) &= \frac{Adj\mathbf{H}(z)}{det\mathbf{H}(z)}\mathbf{t}(z) \end{aligned}$$

Problems:

- $\mathbf{f}(z)$ could be IIR, even if $\mathbf{H}(z)$ is FIR
- $\mathbf{H}(z)$ can be singular
- hard to design $\mathbf{H}(z)$ so that $det\mathbf{H}(z)$ is stable

Polyphase Representation Scheme

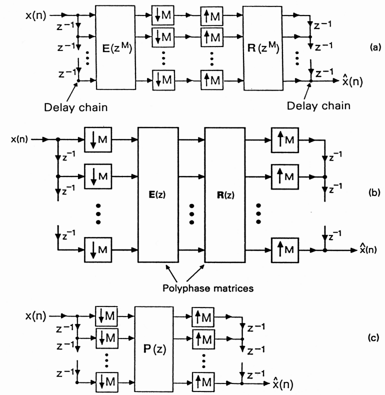


Figure 5.5-3 (a) Polyphase representation of an M-channel maximally decimated filter bank. (b) Rearragement using noble identitites. (c) Further simplification, where $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$.

Polyphase Representation: Analysis Bank

Type 1 Polyphase:

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M)$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0(M-1)}(z^M) \\ E_{10}(z^M) & E_{11}(z^M) & \cdots & E_{1(M-1)}(z^M) \\ \vdots & \vdots & \ddots & \vdots \\ E_{(M-1)0}(z^M) & E_{(M-1)1}(z^M) & \cdots & E_{(M-1)(M-1)}(z^M) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

$$\mathbf{h}(z) = \mathbf{E}(z^M)\mathbf{e}(z)$$

Polyphase Representation: Synthesis Bank

Type 2 Polyphase:

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{kl}(z^M)$$

$$\begin{bmatrix} F_0(z) & F_1(z) & \cdots & F_{M-1}(z) \end{bmatrix} =$$

$$\begin{bmatrix} z^{-(M-1)} & z^{-(M-2)} & \cdots & 1 \end{bmatrix} \begin{bmatrix} R_{00}(z^M) & R_{01}(z^M) & \cdots & R_{0(M-1)}(z^M) \\ R_{10}(z^M) & R_{11}(z^M) & \cdots & R_{1(M-1)}(z^M) \\ \vdots & \vdots & \ddots & \vdots \\ R_{(M-1)0}(z^M) & R_{(M-1)1}(z^M) & \cdots & R_{(M-1)(M-1)}(z^M) \end{bmatrix}$$

$$\mathbf{f}^T(z) = (z^{-(M-1)})\tilde{\mathbf{e}}(z)\mathbf{R}(z^M)$$

Alias Free Systems 1

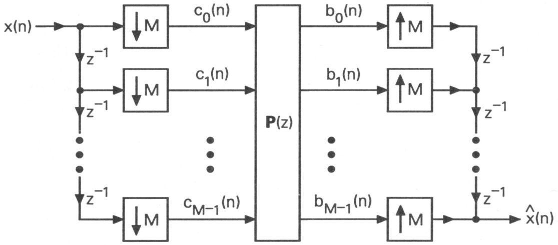


Figure 5.7-3 The equivalent circuit for the maximally decimated filter bank.

$$\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$$

Alias Free Systems 2

$$\tilde{X}(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(zW^k)\sum_{l=0}^{M-1} W^{-kl}\sum_{s=0}^{M-1} z^{-l}z^{-(M-1-s)}P_{s,l}(z^M)$$

$$\tilde{X}(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(zW^k)\sum_{l=0}^{M-1} W^{-kl}V_l$$

Aliascancellation:

$$\sum_{l=0}^{M-1} W^{-kl}V_l = 0 \quad \textit{for all } k \neq 0$$

Alias Free Systems 3

$$\sum_{l=0}^{M-1} W^{-kl} V_l = 0 \quad \textit{for all } k \neq 0$$

$$\mathbf{W}^{\dagger} \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \Omega \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \mathbf{W} \begin{bmatrix} \Omega \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Ω is arbitrary, but unequal zero

Alias Free Systems 4

M=3:

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}2} \\ 1 & e^{-j\frac{2\pi}{3}2} & e^{-j\frac{2\pi}{3}4} \end{bmatrix} \begin{bmatrix} \Omega \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow V_0(z) = V_1(z) = V_2(z) = V(z)$$

Generally:

$$\Rightarrow V_l(z) = V(z) \quad for\ 0 \leq l \leq M-1$$

Alias Free Systems 5: Structure of **P** 1

$$\mathbf{V}_0(\mathbf{z}) = z^{-2}P_{0,0}(z^3) + z^{-1}P_{1,0}(z^3) + z^0\, P_{2,0}(z^3)$$

$$= \mathbf{V}_1(\mathbf{z}) = z^{-3}P_{0,1}(z^3) + z^{-2}P_{1,1}(z^3) + z^{-1}P_{2,1}(z^3)$$

$$= \mathbf{V}_2(\mathbf{z}) = z^{-4}P_{0,2}(z^3) + z^{-3}P_{1,2}(z^3) + z^{-2}P_{2,2}(z^3)$$

$$\begin{aligned}
P_{0,0}(z^3) &= P_{1,1}(z^3) = P_{2,2}(z^3) = P_0(z^3) \\
P_{1,0}(z^3) &= P_{2,1}(z^3) = z^{-3}P_{0,2}(z^3) = z^{-3}P_2(z^3) \\
P_{2,0}(z^3) &= z^{-3}P_{0,1}(z^3) = z^{-3}P_{1,2}(z^3) = z^{-3}P_1(z^3)
\end{aligned}$$

Alias Free Systems 5: Structure of **P** 2

$$\mathbf{P}(z) = \mathbf{E}(z)\mathbf{R}(z) = \begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix}$$

$\Rightarrow$ every alias free system must have a **pseudocirculant** P-matrix in polyphase form

Pseudocirculant Matrix:
every row is a right-shifted copy of the row before,
elements under the main diagonal possess an additional z^{-1}

Perfect Reconstruction Filters: Objectives

- All filters should be FIR (polyphase decomposition is simple, easy linear phase implementation)
- M can be arbitrary
- $H_k(z)$ provides as much attentuation as the user specifies
- Implementation Cost: competitive with approximate reconstruction systems

Perfect Reconstruction Filters: Structure of P

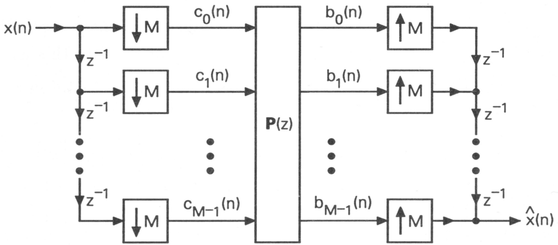


Figure 5.7-3 The equivalent circuit for the maximally decimated filter bank.

Intuitive Solution:

$$\mathbf{P}(z) = c \cdot z^{-m_0}\mathbf{I}, \quad m_0 \geq M \textit{ for causality}$$

Perfect Reconstruction Filters: Structure of P 2

Most General Solution:

$$\mathbf{P}(z) = c \cdot z^{-m_0} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(M-r)\times(M-r)} \\ z^{-1}\mathbf{I}_{r\times r} & \mathbf{0} \end{bmatrix}, \quad 0 \leq r \leq M-1, c \neq 0$$

$$T(z) = cz^{-r} z^{-(M-1)} z^{m_0 M}$$

Perfect Reconstruction Filters: Example

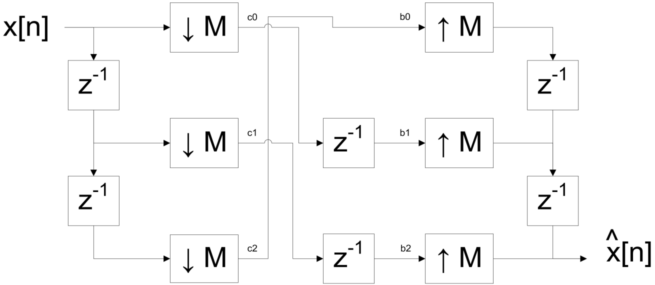
Easiest Configuration:

just delays, M=3

$$\mathbf{P}(z) = \begin{bmatrix} 0 & 0 & 1 \\ z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \end{bmatrix}$$

$$\mathbf{b}(z) = \mathbf{P}(z)\mathbf{c}(z)$$

Perfect Reconstruction Filters: Example 2



...,0,1,2,3,4,5,6,0,...

0,1,2,3,4,5,6
0,0,1,2,3,4,5,6
0,0,0,1,2,3,4,5,6

0,3,6
0,2,5
0,1,4

0,1,4
0,0,3,6
0,0,2,5

0,0,0,1,0,0,4,0,0,0
0,0,0,0,0,0,3,0,0,6
0,0,0,0,0,0,2,0,0,5

0,0,0,0,0,1,0,0,4,0,0
0,0,0,0,0,0,0,3,0,0,6
0,0,0,0,0,0,2,0,0,5,0

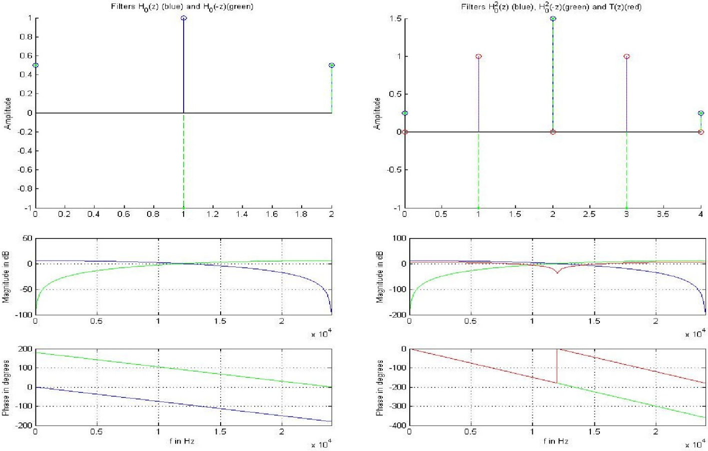
0,0,0,0,0,1,2,3,4,5,6

Summary

These are the mathematical basics,
but there are countless possibilities for the design an Implementation of a filter bank...

IEEE Explorer:
Maximally Decimated Filter Bank: 446 532 documents
QMF Bank: 1 566 306 documents

Appendix: Phase Distortion Example



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